Lemma: Let $M$ be a smooth manifold of dim $n$ (Riemannian not assumed) and let $\nabla$ be an affine connection on $M$.

Let $(U, \varphi)$ be a chart and consider a coordinate frame on the tangent spaces.

Then $\nabla$ is defined by $n^3$ functions

$\Gamma_{i,j}^k : U \longrightarrow \mathbb{R}$ for $i,j,k \in \{1, \cdots, n\}$ called the CHRISTOFFEL SYMBOLS.

Proof: Let $X, Y \in \mathfrak{X}(M)$, and $\forall p \in U$:

$$X_p = \sum_{i=1}^{n} \underbrace{b^i(p)}_{\in\mathbb{R}} E_{i,p} \; ; \; Y_p = \sum_{i=1}^{n} \underbrace{a^i(p)}_{\in\mathbb{R}} E_{i,p} \,.$$

$E_{i,p}$: bases of $T_p(M)$

Set

$$\overbrace{\nabla_{E_{i,p}} E_{j,p}}^{\in T_p(M)} =: \sum_{k=1}^{n} \Gamma_{i,j}^k (p) E_{k,p}$$

Then

$$\nabla_X Y = \nabla_{\sum_i b^i E_i} \sum_j a^j E_j \overset{\text{Linearity and 1)}}{=} \sum_{i,j} b^i \nabla_{E_i} (a^j E_j) \overset{2)}{=} \sum_{i,j} b^i \left\{ (E_i a^j) E_j + a^j \sum_k \Gamma_{i,j}^k E_k \right\}$$

$$= \sum_k \left( X a^k + \sum_{i,j} a^j b^i \Gamma_{i,j}^k \right) E_k \qquad (*)$$

$\Rightarrow \nabla$ can be expressed by $\Gamma_{i,j}^k$.

Conversely, if we start with $(*)$, it defines an affine connection. (5-min exercise) □

Remark: 1) with these notations

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = \sum_{ijk} (\Gamma_{i,j}^k - \Gamma_{j,i}^k) a^i b^j E_k .$$

Thus $\forall X, Y \in \mathfrak{X}(M) : T(X,Y) = 0 \Leftrightarrow \forall i,j,k : \Gamma_{i,j}^k = \Gamma_{j,i}^k$

2) If $(M, \phi)$ is Riemannian, recall that

$$g_{ij}(p) = \phi_p(E_{i,p}, E_{j,p}) \quad \forall i,j \in \{1, \cdots, n\} \quad \text{and then}$$

$$\Gamma_{i,j}^k = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left( \frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{lj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right)$$

$g^{kl}$: inverse matrix of $(g_{ij})$

(proof as exercise)

A new look at the covariant derivative:

Let $c : I \ni t \mapsto c(t) \in M$ be a smooth curve on $M$, and let $Y \in \mathfrak{X}(M)$.

Let $(U, \varphi)$ be a local chart, and for $p \in U$

$$Y_p = \sum_{k=1}^{n} b^k(p) E_{k,p}.$$

Then we set

$$\frac{DY}{dt}(t) := \left[\nabla_{\dot{c}(t)} Y\right]_{c(t)} = \sum_{k=1}^{n} \left( \dot{c}(t)\, b^k(c(t)) + \sum_{i,j} \Gamma_{i,j}^{k}(c(t))\, b^i(c(t))\, \dot{c}^j(t) \right) E_{k, c(t)}$$

$$\dot{c}(t) = \sum_k \dot{c}^k(t)\, E_{k, c(t)}$$

Observe that

$$\dot{c}(t)\, b^k = c_*\left(\frac{d}{dt}\right)\Big|_t b^k = \frac{d}{dt}(b^k \circ c)\Big|_t = \frac{d}{dt} b^k(c(t)).$$

$$= \sum_{k=1}^{n} \left( \frac{d b^k(c(t))}{dt} + \sum_{i,j} \Gamma_{i,j}^{k}(c(t))\, b^i(c(t))\, \dot{c}^j(t) \right) E_{k, c(t)} \qquad \text{(A)}$$

Remark: only the values of $Y$ on the curve are taken into account.

Def. Let $c : I \mapsto M$ be a curve on $M$, and $\nabla$ an affine connection on $M$.

A vector field $Y : I \ni t \mapsto Y(t) \in T_{c(t)}(M)$ is PARALLEL along $c$ if

$$\frac{DY}{dt}(t) = 0 \quad \forall t \in I.$$

Since (A) is a group of first-order differential equations we have:

Prop. 1) Given a smooth curve $c : (-\varepsilon, \varepsilon) \ni t \mapsto c(t) \in M$ and given $Y_{c(0)} \in T_{c(0)}(M)$ then

$$\exists!\, Y : (-\varepsilon, \varepsilon) \ni t \mapsto Y(t) \in T_{c(t)}(M) \text{ parallel to } c.$$

2) If $(M, \phi)$ is a Riemannian manifold and if $\{F_1, \cdots, F_n\}$ is an orthonormal basis of $T_{c(0)}(M)$ then $\exists!$ orthonormal frame at $c(t)$ which is parallel to $c$.

More generally on Riemannian manifolds, parallel transport preserves the length and the inner product.

## IV.3 Geodesics

Let $c: I \mapsto M$ be a curve on $M$ and $\nabla$ be an affine connection.

($\xleftrightarrow{\text{locally}}$ a set of Christoffel's symbols)

Def. $c$ is GEODESIC (with respect to $\nabla$) if $\dot{c}$ is parallel along $c$, which means

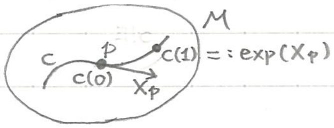
$$\frac{D\dot{c}}{dt}(t) = 0 \quad \forall t \in I$$

$$\Leftrightarrow \ddot{c}^k + \sum_{i,j} \Gamma_{i,j}^k \dot{c}^i \dot{c}^j = 0 \quad \forall k = 1, \cdots, n \quad \text{(geodesic equation)}$$

Remark: since the geodesic equation is a second-order differential equation,

given $p \in M$ and $X_p \in T_p(M)$,

$\exists ! \, c: (-\varepsilon, \varepsilon) \mapsto M$ geodesic s.t. $c(0) = p$ and $\dot{c}(0) = X_p$.




Note that $\forall a > 0$, if we set $c_a: (-\frac{\varepsilon}{a}, \frac{\varepsilon}{a}) \mapsto M$ then

$c_a(0) = p$, $\dot{c}_a(0) = aX_p$ and $c_a$ is again geodesic. Then

Def. $\exp(X_p) := c(1)$ whenever defined. $:\Leftrightarrow \forall u \in U \ \forall a \in [0,1]: au \in U$

Prop. $\forall p \in M \ \exists$ open set $U \subset T_p(M)$ star-shaped with $0 \in U$ s.t.

$\exp: U \mapsto M$ is a diffeomorphism onto $V \subset M$ with $p \in V$.

The proof involves some uniformity.

$\operatorname{Exp}(U)$ is called a NORMAL NEIGHBORHOOD of $p$ on $M$,

and $\exp$ is called the EXPONENTIAL MAP.

Remark: If $(M, \phi)$ is a Riemannian manifold,

and if $\{F_1, \cdots, F_n\}$ is an orthonormal basis of $T_p(M)$, then

$$X_p = \sum_{j=1}^n \underbrace{x^j}_{\in \mathbb{R}} F_j \quad \text{(unique decomposition)}$$

Then

$$\varphi: \exp(U) \ni \exp(\underset{\substack{\cup\\ \mathrm{U}}}{X_p}) = \exp\left(\sum_{j=1}^n x^j F_j\right) \longmapsto (x^1, \cdots, x^n) \in \mathbb{R}^n$$

and $(\exp(U), \varphi)$ is a coordinate system around $p$, called

the NORMAL COORDINATE SYSTEM around $p$.

(with special properties)