Def. Two R_o (Riemannian) manifolds (M_1, ϕ_1) and (M_2, ϕ_2) are ISOMETRIC if
$\exists F : M_1 \mapsto M_2$ a diffeomorphism such that $F^* \phi_2 = \phi_1$
$\Rightarrow d_1(p, q) = d_2(F(p), F(q))$ (distance)

Remark: (Nash embedding thm) asserts that
any R_o manifold can be isometrically embedded in $\mathbb{R}^d$, for $d \geq \frac{n(3n+11)}{2}$.

IV.2 Differentiation

Differentiation is important for the description of an evolution or a transport.

Example: In $\mathbb{R}^3$ for a fixed relavence system, $\dot{x}(t) = v$

One can also consider a moving reference system. (moving frame)

Example: We attach a reference system to a point moving in $\mathbb{R}^3$.

Let $s \mapsto c(s)$ be a curve in $\mathbb{R}^3$, with the arc length parameter.

Set $T(s) := c'(s)$, with the property $\|T(s)\| = 1$.

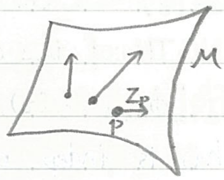
Then $\dot{T}(s) \equiv T'(s) \perp T(s)$ and set $T(s) = K(s) N(s)$ with $K(s) \geq 0$ and $\|N(s)\| =$

the curvature   suppose $K(s) \neq 0$

Consider $\{T(s), N(s), B(s)\}$
as a basis at $c(s)$ orthonormal

The equation of motion of this frame is given by the Sevret-Frenet formula

There are 2 parameters:
$K(s)$ = the curvature
$J(s)$ = the torsion



Example: Let M be a manifold of dim n in $\mathbb{R}^d$.

Let $Z \in \mathfrak{X}(\mathbb{R}^d)$ and let $p \in M$. $\Rightarrow Z_p \in T_p(\mathbb{R}^d)$ but not always $Z_p \in T_p(M)$.

If $Z_p \in T_p(M)$ (tangent to M at p) for any $p \in M$,
we say that Z is a tangent vector field.

Since $\mathbb{R}^d$ has a scalar product, it endows M with a scalar project
$\Rightarrow T_p(\mathbb{R}^d)$ has a scalar product, as well as $T_p(M)$.
$\Rightarrow T_p(\mathbb{R}^d) = T_p(M) \oplus T_p(M)^\perp$
$\Rightarrow \exists \Pi_p$ and $\Pi_p^\perp$ two orthogonal projections on $T_p(M)$ and $T_p(M)^\perp$.

Def. Let $Y \in \mathfrak{X}(M) \subset \mathfrak{X}(\mathbb{R}^d)$ and consider $t \mapsto c(t) \in M \subset \mathbb{R}^d$ a curve on M.

Set $Y(t) := Y_{c(t)} \in T_{c(t)}(M)$ and consider

$$\frac{DY}{dt}(t) := \Pi_{c(t)}\Big(\underbrace{\frac{d}{dt}Y(t)}_{\text{computation in } \mathbb{R}^d}\Big) \in T_{c(t)}(M)$$

called the COVARIANT DERIVATIVE of Y along c.

Thus, Y and $\frac{DY}{dt}$ belong to $\mathfrak{X}(M)$ but the defination of $\frac{DY}{dt}$ uses $\mathbb{R}^d$.

Prop. 1) $\frac{D}{dt}(Y_1 + Y_2) = \frac{DY_1}{dt} + \frac{DY_2}{dt}$

2) $\frac{D}{dt}(fY) = f'Y + f\frac{DY}{dt}$ with any $f \in C^\infty(M)$

$\frac{d}{dt}\langle Y_1, Y_2\rangle = \langle \frac{DY_1}{dt}, Y_2\rangle + \langle Y_1, \frac{DY_2}{dt}\rangle$ with $\boxed{Y_1 = Y_1 \circ c,\ Y_2 = Y_2 \circ c}$

⚠ $\frac{DY}{dt} = 0 \not\Leftrightarrow \frac{dY}{dt} = 0$

— End of example 3

Remark

If we consider $X_p \in T_p(M)$ and

if we choose a curve $t \longmapsto c(t) \in M$ with $c(t_0) = p$ and $\dot{c}(t_0) = X_p$

then $\frac{DY}{dt}(t_0)$ does not depend on $c(t)$ but only on X_p.

(proof as exercise)

It means we can define a map

$$\begin{array}{ccccc} T_p(M) & \times & \mathfrak{X}(M) & \longmapsto & T_p(M) \\ \Downarrow & & \Downarrow & & \Downarrow \\ X_p & & Y & & \frac{DY}{dt}(t_0) =: \nabla_{X_p} Y \end{array}$$

new notation

or more generally

$$\begin{array}{ccccc} \mathfrak{X}(M) & \times & \mathfrak{X}(M) & \longmapsto & \mathfrak{X}(M) \\ \Downarrow & & \Downarrow & & \Downarrow \\ X & & Y & & \nabla_X Y \end{array}$$

with $(\nabla_X Y)_p = \nabla_{X_p} Y$.

Def. An AFFINE CONNECTION on a smooth manifold M is a bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \mathfrak{X}(M)$$
$$X \qquad Y \quad \mapsto \nabla(X,Y) \equiv \nabla_X Y \text{ satisfying}$$

1) $\nabla_{fX} Y = f \nabla_X Y$ $C^\infty(M)$-linearity in the first variable

2) $\nabla_X (fY) = (Xf) Y + f \nabla_X Y$ $\Big\}\ \forall f \in C^\infty(M)$

Def. For any $X, Y \in \mathfrak{X}(M)$ we set

$$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \in \mathfrak{X}(M)$$

called the TORSION of the connection; and set

$$R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \mathrm{End}(\mathfrak{X}(M))$$

$[\tilde{R}(X,Y,Z) := R(X,Y)Z]$ $\quad$ $\hookrightarrow : \mathfrak{X}(M) \longmapsto \mathfrak{X}(M)$; endomorphism 自同態

called the CURVATURE of the connection.

Lemma:

$T(X,Y)$ is $C^\infty(M)$-linear in both arguments;

$\tilde{R}(X,Y,Z)$ is " " in the 3 arguments.

[Exercise; see Tu (geometry) p.44]

Def. On a R_0 manifold, a torsion free ($:\Leftrightarrow T(X,Y)=0\ \forall X,Y$) connection satisfying

$$Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \in C^\infty(M) \quad \forall X,Y,Z \in \mathfrak{X}(M)$$

is called a RIEMANNIAN CONNECTION or LEVI CIVITA CONNECTION.

(compability condition between the Riemannian metric ϕ and the connection ∇)

$\langle X,Y\rangle : M \mapsto \mathbb{R}$; $\langle X,Y\rangle_p := \phi_p(X_p, Y_p) \in \mathbb{R}$

Thm. On a Riemannian manifold $\exists!$ Riemannian connection.

This connection satisfies

$$2\langle \nabla_X Y, Z\rangle = X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle + \langle Z,[X,Y]\rangle$$

(Koszul formula)