Differential Geometry

Extrinsic / Intrinsic ways to study DG (they're not so different)

from outside of the manifold / on the manifold

Extrinsic : to look at curves or surfaces from outside in a bigger space
(in Calculus II) simple for visualization

Intrinsic : no more any ambiant space, like a 2D animal in a flatland ~~witho~~ without a 3^{rd} dimension, useful in general relativity & universe
(mostly used in this course)

However, a manifold can always embedded in a higher dimensional space (not always one more)
(Nash embedding thm)

I) Differentiable manifolds

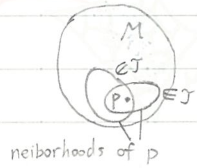
I.1 Topological manifolds (+ topology)

Def. a TOPOLOGICAL MANIFOLD of dimension n is a $\boxed{\text{topological space}}$ $\mathcal{M}$ s.t. :
1) $\mathcal{M}$ is $\boxed{\text{Hausdorff}}$
2) Any $p \in \mathcal{M}$ has a $\boxed{\text{neighborhood}}$ V $\boxed{\text{homeorphic}}$ to an $\boxed{\text{open set}}$ $U \subset \mathbb{R}^n$.
3) $\mathcal{M}$ is $\boxed{\text{second countable}}$.

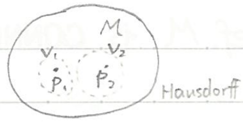
$\boxed{\text{Def}}$ a TOPOLOGICAL SPACE $\mathcal{M} \equiv (\mathcal{M}, \underset{\text{curly T}}{\mathcal{T}})$
is a set $\mathcal{M}$ together with a collection $\mathcal{T}$ of subsets satisfying :
1) $\phi, \mathcal{M} \in \mathcal{T}$ (empty set 空集)
2) If $V_\alpha \in \mathcal{T}$, then $\bigcup_\alpha V_\alpha \in \mathcal{T}$ (union 並集) ($\mathcal{T}$ is STABLE FOR ARBITRARY UNION)
3) If $V_1, \cdots, V_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} V_i \in \mathcal{T}$ (intersection 交集) (↳ UNDER FINITE INTERSECTION)

The elements of $\mathcal{T}$ are called the OPEN SETS.
Their complement $(\mathcal{M} \setminus V,\ V \in \mathcal{T})$ is called a CLOSED SET.

$\boxed{\text{Def}}$ Let $(\mathcal{M}, \mathcal{T})$ be a topological space (t.s.), and let $p \in \mathcal{M}$.
a NEIBORHOOD of p is any open set containing p.
We write $\mathcal{V}_p$ (Greek nu) for the set of all neighborhoods of p.

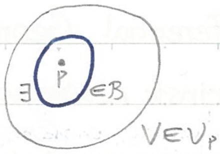


$\boxed{\text{Def}}$ $(\mathcal{M}, \mathcal{T})$ is HAUSDORFF if
$$\forall p_1, p_2 \in \mathcal{M},\ p_1 \neq p_2 : \exists V_1 \in \mathcal{V}_{p_1},\ V_2 \in \mathcal{V}_{p_2} : V_1 \cap V_2 = \phi$$



It is often difficult to describe all open sets in $(\mathcal{M}, \mathcal{T})$
⇒ Introduce the notion of a basis. (related to Second Countable)

Def. A subset $\mathcal{B} := \{V_\alpha\} \subset \mathcal{T}$ is a BASIS of $(M, \mathcal{T})$ if



$$\forall p \in M \ \forall V \in \mathcal{V}_p : \exists U \in \mathcal{B} : p \in U \subset V$$

Example: $M = \mathbb{R}^n$ with $\mathcal{T} = \{$all open sets in $\mathbb{R}^n\}$ is a topological manifold.

An OPEN SET in $\mathbb{R}^n$ is a set V s.t. $\forall p \in V$:

there is a small ball centered at p and contained in V.

We set $B(p, r)$ = a ball centered at p and of radius r.

norm in $\mathbb{R}^n$

$$B(p, r) := \{x \in \mathbb{R}^n \mid \|x - p\| < r\}$$

Then: (all balls centered at any point)

$$\mathcal{B} := \{B(x, r) \mid x \in \mathbb{R}^n, r > 0\} \text{ is a basis for } \mathbb{R}^n.$$

in a 1 to 1 (= bijective) relation with $\mathbb{N}$.

Def. $(M, \mathcal{T})$ is SECOND COUNTABLE if it has a countable basis.

For $\mathbb{R}^n$, we can set

$\mathcal{B} := \{B(x, \frac{1}{m}) \mid x \in \mathbb{Q}^n, m \in \mathbb{N}\}$ and it is a countable basis for $\mathbb{R}^n$.

$\Rightarrow \mathbb{R}^n$ is second countable.

Def. Let $(M, \mathcal{T}), (\mathcal{N}, \mathcal{S})$ be 2 t.s., and let $f: M \mapsto \mathcal{N}$.

f is CONTINUOUS if $f^{-1}(U) \in \mathcal{T} \ \forall U \in \mathcal{S}$

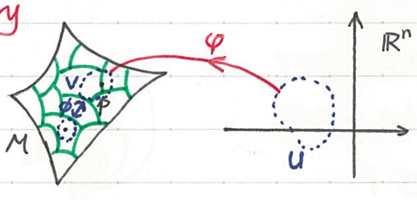
with the PRE-IMAGE $f^{-1}(U) := \{p \in M \mid f(p) \in U\}$

Exersise: When $M = \mathcal{N} = \mathbb{R}$ and $\mathcal{T} = \mathcal{S} = \{$open sets in $\mathbb{R}\}$, check if this def corresponds to the ε-δ def of coutinuousity.

If f is bijective and f, f^{-1} are continuous,

we say that f is HOMEOMORPHIC.

Summary



Def. M is CONNECTED if it is not the disjoint union of 2 non-empty open sets.

Def. Let A be a subset of M.

1) An OPEN COVER for A is a subfamily $\{V_\alpha\} \subset \mathcal{T}$ s.t. $A \in \bigcup_\alpha V_\alpha$ finite or infinite

2) a SUBCOVER of an open cover for A (in which the green subsets are unnecessary) is a subfamily $\{V_\beta\} \subset \{V_\alpha\}$ which still covers A.

3) A is COMPACT (small in this setting) if any open cover of A admits a finite subcover (If $A = \mathbb{R}^n$, A is compact iff A is closed and bounded)

$(\mathbb{N}, \mathcal{T})$ topo. space

$\mathcal{T}_0 := \{[a,b] \cap \mathbb{N} \mid a \text{ is not odd and } b \text{ is not even}; a < b; a, b \in \mathbb{N} \cup \{\infty\}\}$

$\mathcal{T} := \{I \mid I = \bigcup_\alpha I_\alpha, \forall \alpha : I_\alpha \in \mathcal{T}_0\} \cup \{\emptyset\}$

$\mathcal{T} := \{(\bigcup_\alpha [A_\alpha, B_\alpha]) \cap \mathbb{N} \mid \forall \alpha : A_\alpha \text{ is not odd and } B_\alpha \text{ is not even}; A_\alpha < B_\alpha; A_\alpha, B_\alpha \in \mathbb{N} \cup \{\infty\}\}$
$\cup \{\emptyset\}$