

V Curvature

V.1 Several curvatures

Framework: M a smooth manifold with ∇ a connection.

If (M, ϕ) is Riemannian, then ∇ is the Levi Civita connection.

Recall that the curvature R is defined on $X, Y \in \mathfrak{X}(M)$ by

$$R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]} \in \text{End}(\mathfrak{X}(M))$$

$$R(X, Y) : \mathfrak{X}(M) \ni Z \mapsto R(X, Y)Z \in \mathfrak{X}(M)$$

Lemma: If ∇ is torsion free then

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$$

[Bianchi identity; GN p. 125]

True also for Levi Civita connection.

In local coordinates [= with a chart (U, ϕ) and the coordinate frame $\{E_{j,p}\}_j$]

$$R(E_i, E_j)E_k = \sum_l R_{ijk}{}^l E_l$$

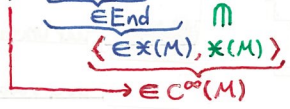
$$\text{with } R_{ijk}{}^l = \frac{\partial}{\partial x^i} \Gamma_{jk}^l - \frac{\partial}{\partial x^j} \Gamma_{ik}^l + \sum_m \Gamma_{jk}^m \Gamma_{im}^l - \sum_m \Gamma_{ik}^m \Gamma_{jm}^l$$

↑ components of R in a basis

⚠ It can be slightly different depending on the authors

For (M, ϕ) , let us also set

$$\phi(R(X, Y)Z, W) := R(X, Y, Z, W) \in C^\infty(M)$$



↑ $\in \mathcal{T}^4(M)$; called the RIEMANNIAN CURVATURE TENSOR

and in local coordinates

$$R_{ijkl} := \phi(R(E_i, E_j)E_k, E_l) = \sum_m R_{ijk}{}^m g_{ml}$$

Lemma: For (M, ϕ)

- 1) $R(X, Y, Z, W) = -R(Y, X, Z, W)$
- 2) $R(X, Y, Z, W) = -R(X, Y, W, Z)$
- 3) $R(X, Y, Z, W) = R(Z, W, X, Y)$

[Exercise; see Boo p. 383 and GN p. 126]

For any $p \in M$, let us denote by Π a PLANE SECTION in $T_p(M)$,
it means Π is a 2D subspace of $T_p(M)$.

Let X_p, Y_p be 2 elements in $T_p(M)$ generating a basis of Π s.t.
 (X_p, Y_p) is an orthonormal basis of Π .

Def. The SECTIONAL CURVATURE $K(\Pi)_p$ of the section Π with basis (X_p, Y_p) is
$$K(\Pi)_p := -R(X_p, Y_p, X_p, Y_p) = -\phi_p(R(X_p, Y_p)X_p, Y_p)$$

Exercise: $K(\Pi)_p$ depends only on the plane Π and not on the choice of a basis.

Thm. For (M, ϕ) with $\dim(M) \geq 3$:

the Riemannian curvature tensor at p is uniquely determined
by the values of all sectional curvatures at p .

[Exercise; see Boo p.385 and GN p.127]

Def.¹⁾ (M, ϕ) is ISOTROPIC at p if

$$K(\Pi)_p = K_p = \text{constant } \forall \Pi;$$

2) (M, ϕ) is ISOTROPIC if it is isotropic at any $p \in M$;

3) If K_p is constant on any $p \in M$, we say that

M has CONSTANT CURVATURE.

Report: manifolds with constant curvature are classified.

Remark: If $\dim(M) = 2$ then M is isotropic, and

$K_p \equiv K(p)$ is called the GAUSS CURVATURE.

Report: on Gauss curvature or on Gauss-Bonnet Thm.

Lemma: If M is isotropic then locally

$$R_{ijkl}(p) = -K_p (g_{ik}g_{jl} - g_{il}g_{jk})(p)$$

Def. The RICCI CURVATURE tensor field

$\text{Ric} \equiv R \equiv S \in J^2(M)$ is defined on $X, Y \in \mathfrak{X}(M)$ by

$$S_p(X_p, Y_p) := \sum_j R(F_{j,p}, X, Y, F_{j,p}) \text{ with } \{F_{j,p}\}; \text{ an orthonormal basis}$$

Remark:¹⁾ It is independent of the choice of a basis of $T_p(M)$.

$$\text{Locally, } S_{ij} = S(E_i, E_j) = \sum_k R_{kij}^k$$

2) The above operation is called a CONTRACTION of a tensor.

If we contract the Ricci curvature we get the SCALAR CURVATURE given

$$S(p) = \sum_j S(F_{j,p}, F_{j,p}) = \sum_{i,j} S_{i,j} g^{ij}(p)$$

The Riemannian curvature tensor,

the sectional curvature,
the Ricci curvature and
the scalar curvature

give some information
on the local structure
of the manifold.