V Curvature

V.1 Several curvatures

Framework: M a smooth manifold with ∇ a connection.

If (M, ϕ) is Riemannian, then ∇ is the Levi Civita connection.

Recall that the curvature R is defined on $X, Y \in \mathfrak{X}(M)$ by

$$R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \mathrm{End}(\mathfrak{X}(M))$$

$$R(X,Y) : \mathfrak{X}(M) \ni Z \longmapsto R(X,Y)Z \in \mathfrak{X}(M)$$

Lemma: If ∇ is torsion free then

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

[Bianchi identity ; GN p. 125]

True also for Levi Civita connection.

In local coordinates [= with a chart (U, ϕ) and the coordinate frame $\{E_{j,p}\}_j$]

$$R(E_i, E_j)E_k = \sum_l R_{ijk}{}^l E_l$$

with $R_{ijk}{}^l = \frac{\partial}{\partial x^i} \Gamma^l_{jk} - \frac{\partial}{\partial x^j} \Gamma^l_{ik} + \sum_m \Gamma^m_{jk} \Gamma^l_{im} - \sum_m \Gamma^m_{ik} \Gamma^l_{jm}$

$R_{ijk}{}^l$: components of R in a basis

⚠ It can be slightly different depending on the authors

For (M, ϕ), let us also set

$$\phi(\underbrace{R(X,Y)}_{\in \mathrm{End}} Z, W) =: R(X,Y,Z,W) \in C^\infty(M)$$

$\langle \in \mathfrak{X}(M), \mathfrak{X}(M) \rangle \longrightarrow \in C^\infty(M)$

$R(X,Y,Z,W) \in \mathcal{T}^4(M)$; called the RIEMANNIAN CURVATURE TENSOR

and in local coordinates

$$R_{ijkl} := \phi(R(E_i, E_j)E_k, E_l) = \sum_m R_{ijk}{}^m g_{ml}$$

Lemma: For (M, ϕ)

1) $R(X,Y,Z,W) = -R(Y,X,Z,W)$

2) $R(X,Y,Z,W) = -R(X,Y,W,Z)$

3) $R(X,Y,Z,W) = R(Z,W,X,Y)$

[Exercise ; see Boo p. 383 and GN p. 126]

For any $p \in M$, let us denote by Π a PLANE SECTION in $T_p(M)$,

it means Π is a 2D subspace of $T_p(M)$.

Let X_p, Y_p be 2 elements in $T_p(M)$ generating a basis of Π s.t. (X_p, Y_p) is an orthonormal basis of Π.

Def. The SECTIONAL CURVATURE $K(\Pi)_p$ of the section Π with basis (X_p, Y_p) is

$$K(\Pi)_p := -R(X_p, Y_p, X_p, Y_p) = -\phi_p(R(X_p, Y_p)X_p, Y_p)$$

Exercise: $K(\Pi)_p$ depends only on the plane Π and not on the choice of a basis.

Thm. For (M, ϕ) with $\dim(M) \geq 3$:

the Riemannian curvature tensor at p is uniquely determined by the values of all sectional curvatures at p.

[Exercise; see Boo p.385 and GN p.127]

Def. 1) (M, ϕ) is ISOTROPIC at p if

$$K(\Pi)_p = K_p = \text{constant} \quad \forall \Pi;$$

2) (M, ϕ) is ISOTROPIC if it is isotropic at any $p \in M$;

3) If K_p is constant on any $p \in M$, we say that M has CONSTANT CURVATURE.

Report: manifolds with constant curvature are classified.

Remark: If $\dim(M) = 2$ then M is isotropic, and

$K_p \equiv K(p)$ is called the GAUSS CURVATURE.

Report: on Gauss curvature or on Gauss-Bonnet Thm.

Lemma: If M is isotropic then locally

$$R_{ijkl}(p) = -K_p(g_{ik}g_{jl} - g_{il}g_{jk})(p)$$

The Riemannian curvature tensor, the sectional curvature, the Ricci curvature and the scalar curvature give some information on the local structure of the manifold.

Def. The RICCI CURVATURE tensor field

$\mathrm{Ric} \equiv R \equiv S \in \mathcal{T}^2(M)$ is defined on $X, Y \in \mathfrak{X}(M)$ by

$S_p(X_p, Y_p) := \sum_j R(F_{j,p}, X, Y, F_{j,p})$ with $\{F_{j,p}\}_j$ an orthonormal basis of $T_p(M)$

Remark: 1) It is independent of the choice of a basis of $T_p(M)$.

Locally, $S_{ij} = S(E_i, E_j) = \sum_k R_{kij}{}^k$

2) The above operation is called a CONTRACTION of a tensor.

If we contract the Ricci curvature we get the SCALAR CURVATURE given

$$S(p) = \sum_j S(F_{j,p}, F_{j,p}) = \sum_{i,j} S_{i,j}\, g^{ij}(p)$$