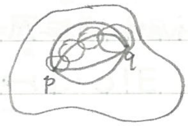
In summary, for a given $p \in M$ $\exists v \in V_p$ (neighborhood) s.t.
any $q \in v$ can be joined to p by a unique geodesic.

With more work one gets

Thm. If c is a piecewise differential path between p and q with
$\underbrace{L(c)}_{\text{length of } c} = \underbrace{d(p,q)}_{\text{distance between } p \text{ and } q \text{ on the Riemannian manifold } M}$ (for defs of L and d, see p.27 in IV.1)
Then c is a geodesic when parametrized by its arc length.

Idea of proof: do it locally.



⚠ The distance is not always realized by a path.

Example: $\mathbb{R}^2 \setminus \{0\}$, $p = (0,1)$, $q = (0,-1)$

Thm. (Hopf and Rinow)
Let (M, ϕ) with Levi-Civita connection ∇.
Are equivalent:

1) exp is defined every on $T_p(M)$ $\forall p \in M$;
2) (M, d) is a COMPLETE metric space (:⇔ with "no holes")
↳ every Cauchy sequence $\subset M$ has a limit $\in M$.
3) Every geodesic $c: I \mapsto M$ can be extended on $\mathbb{R}$.

Def. (M, ϕ) is GEODESICALLY COMPLETE
if one (⇒ all) of these conditions is satisfied.

Lemma. If (M, ϕ) is COMPACT then it is geodesically complete.

Proof: Based on the fact that any compact metric space is complete. □