Proof for Lecture 10

Def. $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \mapsto \mathfrak{X}(M)$ bilinear

$X \quad Y \mapsto \nabla(X,Y) \equiv \nabla_X Y$ satisfying

$$\left.\begin{aligned} &1)\ \nabla_{fX} Y = f \nabla_X Y \\ &2)\ \nabla_X (fY) = (Xf)Y + f\nabla_X Y \end{aligned}\right\} \forall f \in C^\infty(M)$$

$T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y] \in \mathfrak{X}(M)$

$R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \in \mathrm{End}(\mathfrak{X}(M))$

$\tilde{R}(X,Y,Z) := R(X,Y)Z \in \mathfrak{X}(M)$

Co-workers:
Zhang Liyang
Sun Chang
Tomoya
Arata
Kato

Lemma: T and $\tilde{R}$ are $C^\infty(M)$-linear in all arguments.

Proof:

① $T(fX,Y) - fT(X,Y)$

$= \boxed{\nabla_{fX} Y}\boxed{-\nabla_Y (fX)}\boxed{-(fX)Y}\boxed{+Y(fX)}\ \boxed{-f\nabla_X Y}\boxed{+f\nabla_Y X}\boxed{+f(XY)}\boxed{-f(YX)}$

Now we have

$\nabla_{fX} Y - f\nabla_X Y = 0$;

$-\nabla_Y (fX) + f\nabla_Y X = -(Yf)X - f\nabla_Y X + f\nabla_Y X = -(Yf)X$;

$\forall \varphi \in C^\infty(M),\ p \in M,$

$[(fX)Y](\varphi)|_p = (fX)[Y(\varphi)]|_p \underset{\text{Lecture 4}}{\overset{\text{See p.11}}{=\!=}} \overset{\nearrow \in \mathbb{R}}{f(p)} X_p[Y(\varphi)] = f(p)\, XY(\varphi)|_p$

Since φ and p are arbitrary, $(fX)Y = f(XY)$;

$[Y(fX)](\varphi) = Y[(fX)(\varphi)] = Y[f(X\varphi)] \underset{\text{rule}}{\overset{\text{Leibniz's}}{=\!=}} (Yf)(X\varphi) + f(YX(\varphi))$

Since φ is arbitrary, $Y(fX) = (Yf)X + f(YX)$

$\Rightarrow T(fX,Y) - fT(X,Y) = 0 - (Yf)X - 0 + (Yf)X + f(YX) - f(YX) = 0$

$\Rightarrow T(fX,Y) = fT(X,Y)$

② $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = -\nabla_Y X + \nabla_X Y + [Y,X] = -T(Y,X)$

$\Rightarrow T(X,fY) = -T(fY,X) = -fT(Y,X) = fT(X,Y)$

③ $R(fX,Y) - fR(X,Y)$

$= \boxed{\nabla_{fX}\nabla_Y}\boxed{-\nabla_Y \nabla_{fX}}\boxed{-\nabla_{(fX)Y}}\boxed{+\nabla_{Y(fX)}}\ \boxed{-f\nabla_X\nabla_Y}\boxed{+f\nabla_Y\nabla_X}\boxed{+f\nabla_{XY}}\boxed{-f\nabla_{YX}}$

Now we have

$\nabla_{fX}\nabla_Y = f\nabla_X\nabla_Y$;

$\nabla_Y\nabla_{fX} Z = \nabla_Y (f\nabla_X Z) = (Yf)(\nabla_X Z) + f\nabla_Y\nabla_X Z$

Since Z is arbitrary, $\nabla_Y \nabla_{fX} = (Yf)\nabla_X + f\nabla_Y\nabla_X$;

$\nabla_{(fX)Y} = \nabla_{f(XY)} = f\nabla_{XY}$;

$\nabla_{Y(fX)} = \nabla_{\overset{\nearrow \in C^\infty(M)}{(Yf)}X} + \nabla_{f(YX)} = (Yf)\nabla_X + f\nabla_{YX}$

$\Rightarrow R(fX,Y) - fR(X,Y) = 0 - (Yf)\nabla_X - 0 + (Yf)\nabla_X + f\nabla_{YX} - f\nabla_{YX} = 0$

$\Rightarrow R(fX,Y) = fR(X,Y)$

④ $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = -\nabla_Y \nabla_X + \nabla_X \nabla_Y + \nabla_{[Y,X]} = -R(Y,X)$

$\Rightarrow R(X,fY) = -R(fY,X) = -fR(Y,X) = fR(X,Y)$

⑤

$$
\begin{aligned}
& R(X,Y)(fZ) \\
&= \nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{XY}(fZ) + \nabla_{YX}(fZ) \\
&= \nabla_X [f \nabla_Y Z + (Yf)Z] - \nabla_Y [f \nabla_X Z + (Xf)Z] \\
&\quad - [f \nabla_{XY} Z + (XYf)Z] + [f \nabla_{YX} Z + (YXf)Z] \\
&= \boxed{f \nabla_X \nabla_Y Z} \boxed{+ (Xf) \nabla_Y Z} \boxed{+ (Yf) \nabla_X Z} \boxed{+ [X(Yf)] Z} \\
&\quad \boxed{- f \nabla_Y \nabla_X Z} \boxed{- (Yf) \nabla_X Z} \boxed{- (Xf) \nabla_Y Z} \boxed{- [Y(Xf)] Z} \\
&\quad \boxed{- f \nabla_{XY} Z} \boxed{- (XYf) Z} \boxed{+ f \nabla_{YX} Z} \boxed{+ (YXf) Z} \\
&= f(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z) \\
&= f R(X,Y) Z
\end{aligned}
$$

Now separately ①~⑤ have shown that $T, \tilde{R}$ are $C^\infty(M)$-linear in all arguments. □