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Let M be a smooth manifold, $\Lambda^n(M)$ be the alternating n-tensor bundle, $A^n(M)$ be the vector spaces of differential n-forms on M, and $d: A^n(M) \longrightarrow A^{n+1}(M)$ be the exterior derivative.

Def
1. An n-form $\omega: M \to \Lambda^n(M)$ is closed if $d\omega = 0$.
2. An n-form ω is exact if $\omega = d\eta$ for some $\eta: M \to \Lambda^{n-1}(M)$

Rmk. Every exact form is closed since $d^2 = d \circ d = 0$.
However, a closed form is not always be exact.

e.g. 1-form on $\mathbb{R}^2 \setminus \{0\}$. $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2}$
Then $d\omega = 0$ but ω is not exact.

Def. The n-th de Rham cohomology group of M

$$H^n(M) := \{\text{closed } n\text{-forms on } M\} / \{\text{exact } n\text{-forms on } M\}$$

Note that
the set of closed n-forms is equivalent to the kernel of $d: A^n(M) \to A^{n+1}(M)$

the set of exact n-forms is equivalent to the image of $d': A^{n-1}(M) \to A^n(M)$

Then $H^n(M)$ can be written as $H^n(M) = \mathrm{Ker}\, d / \mathrm{Im}\, d'$

※ $H^n(M)$ is a quotient vector space over $\mathbb{R}$

Rmk. (About quotient vector spaces)

Let V, W be vector subspaces on $\mathbb{R}$ (or $\mathbb{C}$)
Suppose that W is a vector subspace of V. Then we shall define a equivalence relation $\sim$ on V.

$$x \sim y \iff x - y \in W \quad (x, y \in V)$$

We set $[x] := \{y \in V \mid x \sim y\} = \{x + w \mid w \in W\}$ (it is called an equivalence class) and
$V/W := \{[x] \mid x \in V\}$. V/W is called a quotient vector space.

$\rightsquigarrow$ $x \sim y \iff [x] = [y]$

Def. Let ω, ω' be closed n-forms.
ω and ω' are called cohomologous if $[\omega] = [\omega']$
i.e. $\omega - \omega'$ is exact.

Prop. Let ω, θ be closed and θ be exact.
Then $\omega \wedge \theta$ is exact.

prf.
θ is exact, so there exists α such that $\theta = d\alpha$.
$\omega \wedge \theta = \omega \wedge d\alpha$.
$d(\omega \wedge \alpha) = \omega \wedge d\alpha = \omega \wedge \theta$, for $d\omega = 0$. $\therefore$ $\omega \wedge \theta$ is exact.
Similarly, $\theta \wedge \omega$ is also exact. $\square$

($\rightsquigarrow$ We can get $\omega \sim \omega'$, $\theta \sim \theta' \Rightarrow \omega \wedge \theta \sim \omega' \wedge \theta'$. Thus we can define $[\omega] \wedge [\theta] := [\omega \wedge \theta]$.)

Prop (Poincaré Lemma)

Let ω be a n-form on $\mathbb{R}^n$ such that $d\omega = 0$. (i.e. ω is closed)

Then there exists a $(n-1)$-form on $\mathbb{R}^n$ η such that $\omega = d\eta$.

Examples

(i) If M is connected, $H^0(M) \simeq \mathbb{R}$

Cor. $H^0(T^2) \simeq \mathbb{R}$ (T^2 = 2-torus)

(ii) $H^p(S^m) \simeq \begin{cases} \mathbb{R} & \text{if } p = 0, m \\ \{0\} & \text{otherwise} \end{cases}$ (S^m : m-sphere)

(iii) $H^p(\mathbb{R}^n) \simeq \{0\}$ $(p > 0)$

$H^0(M) = \dfrac{Z^0(M)}{B^0(M)}$

(i) : the set of closed 0-forms on M, $Z^0(M)$ is the space of functions on M with zero derivative, and the set of exact 0-forms on M, $B^0(M) \simeq \{0\}$. Taking into account that M is connected, any C^∞-function f on M with $df = 0$, is a constant function on M. Therefore, we can say that $H^0(M) \simeq Z^0(M) \simeq \mathbb{R}$ (1-dimensional vector space)

(iii) : We can get this result using Poincaré Lemma. Poincaré Lemma says that every closed p-form on $\mathbb{R}^n$ is exact. Thus, the p-th de Rham cohomology.

$$H^p(\mathbb{R}^n) = \frac{Z^p(\mathbb{R}^n)}{B^p(\mathbb{R}^n)} \simeq \{0\}$$

(ii) : (It is not obvious. The proof is a bit difficult.)

Thm. Let M be a compact manifold. Then, the de Rham cohomology is a vector space with finite dimension.

Remark

- Generally, it is difficult to calculate the de Rham cohomology of a manifold. However, calculating de Rham cohomologies enables us to know whether two manifolds are "different" or not in some sense. This theorem is commonly used.

Thm. Let M, N be diffeomorphic manifolds. Then, the de Rham cohomologies $H^p(M)$ and $H^p(N)$ are isomorphic. (as vector spaces)

(If you want to study more, it might be good to study Mayer-Vietoris sequence. (algebraic geometry))

(Reference)

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