0617002063 Yoshiyuki ENDO

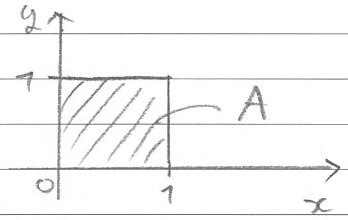
An example of topological manifolds : The torus

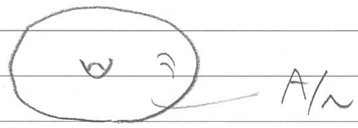
Proposition

Let $A := \{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1 \}$.

If $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in A$, define $a \sim b$ to mean that $a - b = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix} \in \mathbb{Z} \times \mathbb{Z}$;

Then $\sim$ is an equivalence relation, and the quotient set $A/\sim$ is a topological manifold of dimension 2.





Lemma 1

Let $\mathcal{T} := \{ \text{open set in } \mathbb{R}^2 \}$

$\mathcal{T}_A := \{ U \cap A \mid U \in \mathcal{T} \}$: relative topology

$\pi : A \longrightarrow A/\sim$: canonical projection

$\quad x \longmapsto \bar{x}$

$\qquad (\bar{x} = \{ y \mid y \sim x \})$

$\mathcal{T}_{A/\sim} := \{ U \subset A/\sim \mid \pi^{-1}(U) \in \mathcal{T}_A \}$.

Then $(A/\sim, \mathcal{T}_{A/\sim})$ is a topological space.

Proof

(i) $\phi, A/\sim \in \mathcal{T}_{A/\sim}$:

Since $\phi = \pi^{-1}(\phi) \in \mathcal{T}_A$, $\phi \in \mathcal{T}_{A/\sim}$.

Since $A = \pi^{-1}(A/\sim) \in \mathcal{T}_A$, $A/\sim \in \mathcal{T}_{A/\sim}$.

(ii) If $U_\lambda \in \mathcal{T}_{A/\sim}$ for any $\lambda \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}_{A/\sim}$:

Let $\{ U_\lambda \}_{\lambda \in \Lambda} \subset \mathcal{T}_{A/\sim}$.

For any $\lambda \in \Lambda$, $\pi^{-1}(U_\lambda) \in \mathcal{T}_A$, therefore $\bigcup_{\lambda \in \Lambda} \pi^{-1}(U_\lambda) \in \mathcal{T}_A$.

Then $\pi^{-1}\left(\bigcup_{\lambda \in \Lambda} U_\lambda \right) = \bigcup_{\lambda \in \Lambda} \pi^{-1}(U_\lambda) \in \mathcal{T}_A$,

and we obtain $\bigcup_{\lambda \in \Lambda} U_\lambda \in \mathcal{T}_{A/\sim}$.

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(iii) If $U_1, \dots, U_n \in \tau_{A/\sim}$, then $\bigcap_{i=1}^{n} U_n \in \tau_{A/\sim}$;

Let $\{U_i\}_{i=1,\dots,n} \subset \tau_{A/\sim}$.

For any $i = 1, \dots, n$, $\pi^{-1}(U_i) \in \tau_A$,

therefore $\bigcap_{i=1}^{n} \pi^{-1}(U_i) \in \tau_A$.

Then $\pi^{-1}(\bigcap_{i=1}^{n} U_i) = \bigcap_{i=1}^{n} \pi^{-1}(U_i) \in \tau_A$, and we obtain $\bigcap_{i=1}^{n} U_i \in \tau_{A/\sim}$.

□

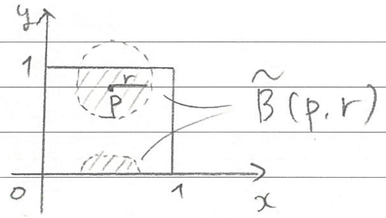
Remark

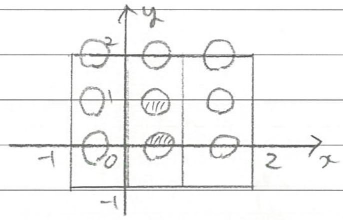
Let $p \in A$, $r \in (0, \frac{1}{2})$.

$X := \left\{ \begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}-1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix} \right\}$.

We set

$$\tilde{B}(p,r) = \bigcup_{\vec{x} \in X} \Big(B(p,r) + \underbrace{\vec{x}}_{\text{translation}} \Big) \cap A$$





Lemma 2

$A/\sim$ is Hausdorff.

i.e. $\forall \bar{p_1}, \bar{p_2} \in A/\sim$, $\bar{p_1} \neq \bar{p_2}$, $\exists V_1$: neighborhood of $\bar{p_1}$, and V_2 : neighborhood of $\bar{p_2}$; $V_1 \cap V_2 = \phi$

Proof

Let $\bar{p_1}, \bar{p_2} \in A/\sim$, $\bar{p_1} \neq \bar{p_2}$.

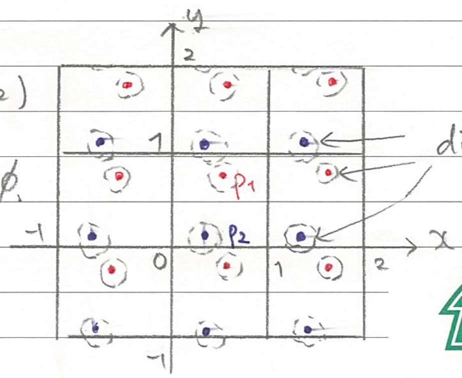
Since $\bar{p_1} \neq \bar{p_2}$, $p_1 \not\sim p_2$.

We set $r' = \min_{\vec{x} \in X} \| p_1, p_2 + \vec{x} \| > 0$, $r = \frac{r'}{3}$.

Now $\pi(\tilde{B}(p_i, r))$ is a neighborhood of $\bar{p_i}$. $(i = 1, 2)$

and $\pi(\tilde{B}(p_1, r)) \cap \pi(\tilde{B}(p_2, r)) = \phi$.

Therefore $A/\sim$ is Hausdorff.



□

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Lemma 3

Any $\bar{p} \in A/\sim$ has a neighborhood V homeomorphic to an open set $U \subset \mathbb{R}^2$.

Proof

Let $\bar{p} \in A/\sim$.

Then $p \in \bar{p}$.

We set $V := \pi(\hat{B}(p,r))$, $U := B(p,r) \subset \mathbb{R}^2$.

Then V is a neighborhood of $\bar{p}$, and U is open.

We define $f : U \to V$ by

$$f(x) = \pi\left(\bigcup_{\vec{x} \in X} \{x + \vec{x}\} \cap A \right).$$

Then $f : U \to V$ is bijective and, f, f^{-1} is continuous. Therefore, V is homeomorphic to U. □

Lemma 4

$A/\sim$ is second countable.

i.e. $A/\sim$ has a countable basis.

Proof

Since A is second countable, it has a countable basis $\mathcal{B}'$.

We set $\mathcal{B} := \{\pi(B)\}_{B \in \mathcal{B}'}$.

Then $\mathcal{B}$ is countable.

$\mathcal{B}$ is a basis of $A/\sim$;

∵ Let $\bar{p} \in A/\sim$, V: neighborhood of $\bar{p}$.

Then $\pi^{-1}(V)$ is a neighborhood of p.

Since $\mathcal{B}'$ is a basis of A, there exists $B \in \mathcal{B}'$ such that $p \in B \subset \pi^{-1}(V)$.

Then $\bar{p} \in \pi(B) \subset V$.

Therefore $A/\sim$ is second countable. □

Summary

By lemma (1) (2) (3) and (4), $A/\sim$ is a topological manifold of dimention 2.