0617002663 Yoshiyuki ENDO

Def Let $(M, \mathcal{T})$, $(N, \mathcal{S})$ be two topological spaces, and let $f: M \to N$.
f is continuous if $f^{-1}(U) \in \mathcal{T}$ for all $U \in \mathcal{S}$
with the pre-image $f^{-1}(U) := \{ p \in M \mid f(p) \in U \}$

Exercise when $M = N = \mathbb{R}$ and $\mathcal{T} = \mathcal{S} = \{$ open sets in $\mathbb{R} \}$, check if this definition corresponds to the ε-δ definition of continuity.

⚠ **Def** an open set in $\mathbb{R}$ is a set V s.t. $\forall p \in V$. there exists a small ball centered at p and contained in V

Let $f: \mathbb{R} \to \mathbb{R}$.

(A) $\forall U \underset{\text{open}}{\subset} \mathbb{R}$. $f^{-1}(U)$: open

(B) $\forall a \in \mathbb{R}$. $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

proof (A) $\Rightarrow$ (B)

Let $a \in \mathbb{R}$, $\varepsilon > 0$, and. $U := (f(a) - \varepsilon, f(a) + \varepsilon)$.
Then U is open, so by (A), $f^{-1}(U)$ is also open.
Now $f(a) \in U$, so $a \in f^{-1}(U)$.
Because $f^{-1}(U)$ is open, there exists $\delta > 0$ such that
$B(a, \delta) := \{ x \in \mathbb{R} \mid \|x - a\| < \delta \} \subset f^{-1}(U)$.
Then $x \in B(a, \delta) (\Leftrightarrow |x - a| < \delta)$

$$\Rightarrow x \in f^{-1}(U)$$
$$\Rightarrow f(x) \in U = (f(a) - \varepsilon, f(a) + \varepsilon)$$
$$\Rightarrow |f(x) - f(a)| < \varepsilon.$$

Therefore we obtain (B).

No.

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(B) $\Rightarrow$ (A)

Let U be an open set in $\mathbb{R}$.

If $a \in f^{-1}(U)$, then $f(a) \in U$.

Then there exists $\varepsilon > 0$ such that $B(f(a), \varepsilon) \subset U$, because U is open.

For this $\varepsilon > 0$, there exists $\delta > 0$ such that
$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$ by (B)

Now $B(a, \delta) \subset f^{-1}(U)$.

$\because$ $y \in B(a, \delta) \Rightarrow |y - a| < \delta$
$\Rightarrow |f(y) - f(a)| < \varepsilon$
$\Rightarrow f(y) \in B(f(a), \varepsilon) \subset U$
$\Rightarrow f(y) \in U$
$\Rightarrow y \in f^{-1}(U)$

Therefore we obtain (A). $\square$