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Exercise If $\dim V = n$, then $\dim V^* = n$.

proof

Let $\{a_1, \cdots, a_n\}$ be a basis for V, and let us define $f_i^* \in V^*$ for each $i = 1, \cdots, n$ by

$$f_i^*(c_1 a_1 + \cdots + c_n a_n) = c_i.$$

Then, for any $h \in V^*$,

$$h(c_1 a_1 + \cdots + c_n a_n) = \sum_{i=1}^{n} c_i h(a_i)$$
$$= \sum_{i=1}^{n} h(a_i) f_i^*(c_1 a_1 + \cdots + c_n a_n).$$

Therefore $h = \sum_{i=1}^{n} h(a_i) f_i^*$,

and $V^* = \mathrm{Span}(f_1^*, \cdots, f_n^*)$ —— (i)

Next, let $p = d_1 f_1^* + \cdots + d_n f_n^* = 0$.

Then, $p(a_i) = d_i = 0$ for all $i = 1, \cdots, n$.

Therefore $\{f_1^*, \cdots, f_n^*\}$ is linear independant. —— (ii)

By (i) and (ii), $\{f_1^*, \cdots, f_n^*\}$ is a basis for V^*.

Therefore $\dim V^* = n$

$\square$