Dam Truyen Duc - NUID: 06 18 018 76

$\boxed{\text{II.1}}$ Tensors

<u>Proposition</u>: For a finite dimensional vector space V, its dual V^* has the same dimensions as V

$$\dim V = \dim V^* = n$$

$\triangledown$ + Let $B = \{e_1, \dots, e_n\}$ be a basis of V

$B^* = \{e^1, \dots, e^n\}$ be a set of linear maps satisfying

$$e^i(e_j) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$

(Kronecker delta)

+ Let $w \in V^*$; $v \in V$

$$\Rightarrow v = \sum_{j=1}^{n} v_j e_j$$

$$\Rightarrow w(v) = w\left(\sum_{j=1}^{n} v_j e_j\right)$$

$$= \sum_{j=1}^{n} v_j w(e_j) \quad (w \text{ is a linear map})$$

We note that $e^i(v) = e^i\left(\sum_{j=1}^{n} v_j e_j\right) = \sum_{j=1}^{n} v_j e^i(e_j)$

$$= v_i$$

$\Rightarrow$ Let $w(e_j) = \lambda_j \in \mathbb{R}$

$$w(v) = \sum_{j=1}^{n} v_j w(e_j)$$

$$= \sum_{j=1}^{n} e^j(v) \lambda_j = \sum_{j=1}^{n} \lambda_j e^j(v)$$

Since the choice of $v \in V$ is arbitrary, we conclude that any linear map $w \in V^*$ can be expressed as a linear combination of $\{e^1, \dots, e^n\}$ (1)

+ Let $v \in V$. Consider the zero map as a linear combination of $\{e^1, \dots, e^n\}$

$$0 = \sum_{i=1}^{n} c_i e^i \; ; \; 0(v) = \sum_{i=1}^{n} c_i e^i(v)$$

Choose $v = e_j$

$$\Rightarrow 0(e_j) = \sum_{i=1}^{n} c_i e^i(e_j) = c_i$$

(by definition of e^i)

$\Rightarrow \forall i \in \mathbb{N}; i \in [1, n]$, $c_i = 0$.
$\Rightarrow \{e^1, \dots, e^n\}$ are linearly independent (2)

From (1) and (2), $\{e^1, \dots, e^n\}$ is basis of V^*
$\Rightarrow$ $\dim (V^*) = n = \dim V$ $\square$

Proposition: V is an n-dimensional vector space over field F, and let W be an m-dimensional vector space over F. Then $\mathcal{L}(V, W)$ (the space of linear transformation from V into W) has dimension $m \times n$

∇ + Let $\mathcal{B} = \{\alpha_1, \dots, \alpha_n\}$ be a basis of V
$\mathcal{B}' = \{\beta_1, \dots, \beta_m\}$ be a basis of W

+ We define an (unique) linear transformation $E^{p,q}$ from V into W: $(1 \leq p \leq m;\ 1 \leq q \leq n)$

$$E^{p,q}(\alpha_i) = \delta_{iq} \beta_p = \begin{cases} 0 \text{ if } i \neq q \\ \beta_p \text{ if } i = q \end{cases}$$

+ Let $T \in \mathcal{L}(V, W)$

$$\Rightarrow T(\alpha_j) = \sum_{p=1}^{m} A_{pj} \beta_p \qquad (1 \leq j \leq n,\ A_{pj} \in \mathbb{R})$$

We note that $\beta_p = \sum_{q=1}^{n} E^{p,q}(\alpha_j)$

$$\Rightarrow \sum_{q=1}^{n} A_{pq} E^{p,q}(\alpha_j) = A_{pj} \beta_j$$

$$\Rightarrow T(\alpha_j) = \sum_{p=1}^{m} A_{pj} \beta_j = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{p,q}(\alpha_j)$$

Since for $\forall v \in V$, $T(v) = T(\sum_{j=1}^{n} v_j \alpha_j) = \sum_{j=1}^{n} v_j T(\alpha_j)$

$\Rightarrow E^{p,q}$ span $\mathcal{L}(V, W)$ $(1 \leq p \leq n;\ 1 \leq q \leq m)$ (1)

+ Consider the zero transformation

$$\bar{0} = \sum_{p=1}^{n} \sum_{q=1}^{m} A_{pq} E^{p,q}$$

Dam Truyen Duc - NUID: 061801876

$\Rightarrow \bar{O}(\alpha_j) = \sum_{p=1}^{n} \sum_{q=1}^{m} A_{pq} E^{p,q}(\alpha_j) = \sum_{p=1}^{m} A_{pj} \cdot \beta_p = 0$

Because of the independence of β_p in the basis of W

$\rightarrow A_{pj} = 0$ for $\forall p \in [1; n]$ and $\forall j \in [1; m]$

$\Rightarrow \{E^{p,q}\}$ are linearly independent (2)

+ From (1) and (2) $\Rightarrow \{E^{p,q}\}$ is a basis of $\mathcal{L}(V, W)$

Since $1 \leq p \leq n$; $1 \leq q \leq m$

$\Rightarrow \dim \mathcal{L}(V, W) = m \times n$.

Proposition The set $\mathcal{T}^r$ (or $\mathcal{T}^r_0$) of all tensors order $(r, 0)$ on V form a vector space of dimension n^r, with $\dim V = n$.

∇ + Let $\{e_1, e_2, \dots, e_n\}$ be the basis of V

$\rightarrow$ For $v_j \in V$

$v_j = \sum_{i=1}^{n} \alpha_j^i e_i$

+ For $\Phi \ni \mathcal{T}^r : V \times V \times \dots \times V \longmapsto \mathbb{R}$

$\Phi(v_1, \dots, v_r) = \Phi(\sum_{i=1}^{n} \alpha_1^i e_i; \sum_{i=1}^{n} \alpha_2^i e_i; \dots; \sum_{i=2}^{n} \alpha_r^i e_i)$

Since Φ is a multilinear map

$\Rightarrow \Phi(v_1, \dots, v_r) = \sum_{i_1, \dots, i_r} \alpha_1^{i_1} \alpha_2^{i_2} \dots \alpha_r^{i_r} \Phi(e_{i_1}, e_{i_2}, \dots, e_{i_r})$

for all sets of $\{i_1, \dots, i_r \mid 1 \leq i_1, \dots, i_r \leq n\}$ natural numbers.

+ Let $\Omega^{i_1, \dots, i_r} \in \mathcal{T}^r$ defines by:

$$\Omega^{i_1, \dots, i_r}(e_{k_1}, \dots, e_{k_r}) = \begin{cases} 1 \text{ if } i_j = k_j \text{ for } 1 \leq j \leq r \\ 0 \text{ if } \exists j \in [1; r] \mid i_j \neq k_j \end{cases}$$

+ For all set of $\{v_1, \dots, v_r \mid \forall v_i \in V\}$ and $\Phi \in \mathcal{T}^r$

$\Rightarrow \Phi(v_1, \dots, v_r) = \sum_{i_1, \dots, i_r} \alpha_1^{i_1} \alpha_2^{i_2} \dots \alpha_r^{i_r} \Phi(e_{i_1}, \dots, e_{i_r})$

Let $\varphi_{i_1, \dots, i_r} = \Phi(e_1, \dots, e_r) \in \mathbb{R}$

Since $\alpha_1^{i_1} \alpha_2^{i_2} \dots \alpha_r^{i_r} = \Omega^{i_1, \dots, i_r}(\alpha_1^{i_1} e_{i_1}, \dots, \alpha_r^{i_r} e_{i_r})$

Using the definition of $\Omega^{i_1,\dots,i_r}$

$\Rightarrow \Omega^{i_1,\dots,i_r}(v_1,\dots,v_r) = \sum_{i_1,\dots,i_r} \alpha_1^{j_1}\cdots\alpha_r^{j_r}\,\Omega^{i_1,\dots,i_r}(e_{j_1},\dots,e_{j_r})$

$= \alpha_1^{i_1}\cdot\dots\cdot\alpha_r^{i_r}$

$\Rightarrow \Phi(v_1,\dots,v_r) = \sum_{i_1,\dots,i_r} \varphi_{i_1\dots i_r}\cdot\Omega^{i_1,\dots,i_r}(v_1,\dots,v_r)$

$\Rightarrow$ span of $\Omega^{i_1,\dots,i_r}$ is $\mathcal{T}^r$, with $1\le i_1,\dots,i_r\le n$ (1)

+ Consider a zero multilinear map $\bar{0}\in\mathcal{T}^r$
and the combination for $v_1,\dots,v_r\in V$

$$\sum_{i_1\dots i_r}\beta_{i_1\dots i_r}\,\Omega^{i_1\dots i_r}(v_1,\dots,v_r) = \bar{0}(v_1,\dots,v_r)$$

As proven above $\Rightarrow \bar{0}(v_1,\dots,v_r) = \sum_{i_1,\dots,i_r}\beta_{i_1\dots i_r}\,\alpha_1^{i_1}\cdot\dots\cdot\alpha_r^{i_r} = 0$

In order to be true for any choice of $v_1,\dots,v_r$ (any choice of α_k^i)

$\Rightarrow \beta_{i_1,\dots,i_r} = 0$ for $1\le i_1,\dots,i_r\le n$

$\Rightarrow \{\Omega^{i_1,\dots,i_r}\}$ are linearly independent. (2)

+ From (1) and (2)

$\Rightarrow \{\Omega^{i_1,\dots,i_r}\}_{1\le i_1,\dots,i_r\le n}$ is a basis of $\mathcal{T}^r$

$\Rightarrow \dim\mathcal{T}^r = n^r$. $\square$

(*) Extension: $\dim\mathcal{T}^r_s = n^{r+s}$

▽ I define $\mathcal{T}^r_s \ni \Omega^{i_1,\dots,i_r,I_1,\dots,I_s}$,

$$\Omega^{i_1,\dots,i_r,I_1,\dots,I_s}(e_{k_1},\dots,e_{k_r}, w^{K_1},\dots,w^{K_s}) = \begin{cases} 1 & \text{if } i_j = k_j \text{ and } I_j = K_j\ (\forall j\in[1,n] \\ 0 & \text{if } \exists j\in[1,n] \mid \left[\begin{matrix} k_j\ne i_j \\ K_j\ne I_j\end{matrix}\right.\end{cases}$$

with $\{e_1,\dots,e_n\}$
and $\{w^1,\dots,w^n\}$ be bases of V and V^*

+ We prove that span $\Omega^{i_1,\dots,i_r,I_1,\dots,I_r}$ is $\mathcal{T}^r_s$

$\mathcal{T}^r_s \ni \Phi(v_1,\dots,v_r; f_1,\dots,f_s) = \Phi\left(\sum_{i=1}^n\alpha_1 e_i;\dots;\sum_{i=1}\alpha_r e_i;\ \sum_{j=1}^n\beta_1^j w^j,\dots,\sum_{j=1}\beta_s^j w^j\right)$
with $\{v_1,\dots,v_r\}\in V^n$; $\{f_1,\dots,f_s\}\in V^*$

$= \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}}\alpha_1^{i_1}\dots\alpha_r^{i_r}\,\beta_1^{j_1}\dots\beta_s^{j_s}\,\Phi(e_{i_1},\dots,e_{i_r}, w^{i_1},\dots,w^{j_s})$

By definition of $\Omega^{i_1,\dots,i_r}_{I_1,\dots,I_s}$

$\Omega^{i_1,\dots,i_r,I_1,\dots,I_s}(v_1,\dots,v_r,f_1,\dots,f_s) = \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}}\alpha_1^{i_1}\dots\alpha_r^{i_r}\beta_1^{j_1},\dots,\beta_s^{j_s}\,\Omega^{i_1,\dots,i_r,I_1,\dots,I_s}(e_{i_1},\dots,e_{i_r}, w^{j_1},\dots,w^{j_s})$

$= \alpha_1^{i_1}\cdot\dots\cdot\alpha_r^{i_r}\cdot\beta_1^{j_1}\cdot\dots\cdot\beta_s^{j_s}$

$\Rightarrow \mathcal{T}^r_s$ is span of $\Omega^{i_1,\dots,i_r,j_1,\dots,j_s}$ $(1\le i_1,\dots,i_r,j_1,\dots,j_s\le n)$ (*)

+ $\sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}}\gamma_{\substack{i_1\dots i_r\\ j_1\dots j_s}}\,\Omega^{i_1,\dots,i_r,j_1,\dots,j_s}(v_1,\dots,v_r,f_1,\dots,f_s) = 0$

$= \sum_{\substack{i_1,\dots,i_r\\ j_1,\dots,j_s}}\gamma_{i_1,\dots,i_r,j_1,\dots,j_s}\,\alpha_1^{i_1}\dots\beta_s^{j_s}$

$\Rightarrow \gamma_{i_1,\dots,i_r,j_1,\dots,j_s} = 0$ in order to be true for all sets of $\{v_1,\dots,v_r\}^*$, $\{f_1,\dots,f_s\}$ (**)

(*) & (**) $\Rightarrow \Omega^{i_1,\dots,i_r}_{j_1,\dots,j_s}$ is basis of $\mathcal{T}^r_s$ $\Rightarrow \dim\mathcal{T}^r_s = n^{r+s} = n^r\cdot n^s$