
Homework 10

Exercise 1 Write out the lower and the upper Riemann sums for the function $x \mapsto x^2$ in the interval $[0, 2]$. Use a regular partition of the interval divided into n subintervals of the same length. The following formula can be used:

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

What happens when $n \rightarrow \infty$?

Exercise 2 Consider the function $[0, 1] \ni x \mapsto e^x \in \mathbb{R}$, and consider a regular partition of $[0, 1]$ divided into n intervals of length $\frac{1}{n}$. Compute the following Riemann sums:

1. $I_l := \sum_{j=0}^{n-1} \frac{1}{n} e^{\frac{j}{n}}$ left rule,
2. $I_r := \sum_{j=1}^n \frac{1}{n} e^{\frac{j}{n}}$ right rule,
3. $I_m := \sum_{j=0}^{n-1} \frac{1}{n} e^{\frac{j+1/2}{n}}$ midpoint rule,
4. $I_{tri} := \frac{1}{2}(I_l + I_r)$ trapezoidal rule,
5. $I_{Sim} := \sum_{j=0}^{n/2-1} \frac{1}{3n} (e^{\frac{2j}{n}} + 4e^{\frac{2j+1}{n}} + e^{\frac{2j+2}{n}})$ for n even Simpson's rule.

Can you illustrate these rules on a drawing and compare their rate of convergence to their limit as $n \rightarrow \infty$? The following formula can be used for any $a > 0$ with $a \neq 1$:

$$\sum_{k=0}^{m-1} a^k = \frac{a^m - 1}{a - 1}.$$

Exercise 3 With Riemann sums, compute the following integral: $\int_0^3 (x^3 - 6x) dx$. You can use the two equalities:

$$\sum_{k=1}^n k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^n k^3 = \left(\frac{n(n+1)}{2} \right)^2.$$

Exercise 4 (Mean value theorem for integrals) Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Show that there exists $c \in (a, b)$ such that

$$\int_a^b f(x) dx = f(c)(b - a).$$

Provide a geometric interpretation of this equality when f is a positive function.