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Lemma: If $\phi \in \Lambda^r(V)$ and $\psi \in \Lambda^s(V)$ then $\phi \wedge \psi = (-1)^{rs} \psi \wedge \phi$

Proof: Let $\{E_1, \dots, E_n\}$ be a basis of V.

$$\phi \in \Lambda^r(V) \Rightarrow \phi = \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} a_{i_1, \dots, i_r} E_{i_1} \wedge \dots \wedge E_{i_r}$$

$$\psi \in \Lambda^s(V) \Rightarrow \psi = \sum_{1 \leq j_1 < j_2 < \dots < j_s \leq n} b_{j_1, \dots, j_s} E_{j_1} \wedge \dots \wedge E_{j_s}$$

$$\Rightarrow \phi \wedge \psi = \sum_{\substack{1 \leq i_1 < \dots < i_r \leq n \\ 1 \leq j_1 < \dots < j_s \leq n}} a_{i_1, \dots, i_r} b_{j_1, \dots, j_s} E_{i_1} \wedge \dots \wedge E_{i_r} \wedge E_{j_1} \wedge \dots \wedge E_{j_s}$$

$$= \sum_{\substack{1 \leq i_1 < \dots < i_r \leq n \\ 1 \leq j_1 < \dots < j_s \leq n}} a_{i_1, \dots, i_r} b_{j_1, \dots, j_s} (-1)^r E_{j_1} \wedge E_{i_1} \wedge \dots \wedge E_{i_r} \wedge E_{i_2} \wedge \dots \wedge E_{j_s}$$

r permutations of E_{j_1} with $E_{i_1}, \dots, E_{i_r}$

$$= \sum_{\substack{1 \leq i_1 < \dots < i_r \leq n \\ 1 \leq j_1 < \dots < j_s \leq n}} a_{i_1, \dots, i_r} b_{j_1, \dots, j_s} \underbrace{(-1)^r (-1)^r \dots (-1)^r}_{s \text{ terms}} E_{j_1} \wedge \dots \wedge E_{j_s} \wedge E_{i_1} \wedge \dots \wedge E_{i_r}$$

$$= (-1)^{rs} \psi \wedge \phi$$

$$\Rightarrow \phi \wedge \psi = (-1)^{rs} \psi \wedge \phi \qquad \square$$