Let V be a n-dimensional real vector space with basis $\{e_1, \dots, e_n\}$, V^* be it's dual space with corresponding dual basis $\{e^1, \dots, e^n\}$.

Propostion. $V^{**} \cong V$

Proof. By definition, V^{**} consists of all linear maps from $V^* \to \mathbb{R}$.

Given $v \in V$, define $\phi(v) \in V^{**}$ by $\phi(v)(w) = w(v)$. Note that $\phi(v)(aw+bu) = (aw + bu)(v) = aw(v) + bu(v) = a\phi(v)(w) + b\phi(v)(u)$, so ϕ is indeed a homorphism.

Now, given $\beta \in V^{**}$, define $\phi^{-1}(\beta)$ by $\phi^{-1}(\beta) = \sum_{i=1}^{n} \beta(e^i)e_i \in V$. Again note that $\phi^{-1}(a\alpha + b\beta) = \sum_{i=1}^{n}(a\alpha + b\beta)(e^i)e_i = a\sum_{i=1}^{n} \alpha(e^i)e_i + b\sum_{i=1}^{n} \beta(e^i)e_i$, so ϕ^{-1} is also a homomorphism.

Moreover, observe that

- For $\beta \in V^{**}$,

$$\begin{aligned}
\phi(\phi^{-1}(\beta))(w) &= \phi(\sum_{i=1}^{n} \beta(e^i)(e_i))(w) \\
&= \sum_{i=1}^{n} \beta(e^i)\phi(e_i)(w) \\
&= \sum_{i=1}^{n} \beta(e_i)w(e^i) \\
&= \beta(\sum_{i=1}^{n} w(e_i)e^i) \\
&= \beta(w)
\end{aligned}$$

  Thus $\phi(\phi^{-1}(\beta)) = \beta$.

- For $v = \sum_{i=1}^{n} v_i e_i \in V$,

$$\begin{aligned}
\phi^{-1}(\phi(v)) &= \sum_{i=1}^{n} \phi(v)(e^i)e_i \\
&= \sum_{i=1}^{n} e^i(v)e_i \\
&= \sum_{i=1}^{n} v_i e_i \\
&= v
\end{aligned}$$

  Thus $\phi^{-1}(\phi(v)) = v$.

Hence ϕ does indeed have an inverse and so is an isomorphism, thus V and V^{**} are indeed isomorphic. □

Let $\mathcal{T}_r^s(V)$ denote the set of $r \times s$ tensors on V.

Propostion. $\mathcal{T}_r^s$ is a real vector space of dimension n^{r+s}.

Proof. Let $\phi \in \mathcal{T}_r^s(V)$, and let $v_1, \ldots, v_r \in V, w_1, \ldots, w_s \in V^*$, then we may write

$$\phi(v_1, \ldots, v_r, w_1, \ldots, w_s) = \phi(\sum_{i=1}^{n} \alpha_{1_i} e_i, \ldots, \sum_{i=1}^{n} \alpha_{r_i} e_i, \sum_{i=1}^{n} \beta_{1_i} e^i, \ldots, \sum_{i=1}^{n} \beta_{s_i} e^i)$$
$$= \sum_{(i_1,\ldots,i_r,j_1,\ldots,j_s)\in\{1,\ldots,n\}^{r+s}} \alpha_{1_{i_1}} \ldots \alpha_{r_{i_r}} \beta_{1_{j_1}} \ldots \beta_{s_{j_s}} \phi(e_{i_1}, \ldots, e_{i_n}, e^{j_1}, \ldots, e^{j_s})$$

Now define $T_{i_1,\ldots,i_r,j_1,\ldots,j_s} \in \mathcal{T}_r^s(V)$ by

$$T_{i_1,\ldots,i_r,j_1,\ldots,j_s}(e_{I_1}, \ldots, e_{J_s}) = \begin{cases} 1 & \text{if } i_1 = I_1, \ldots, j_s = J_s \\ 0 & \text{otherwise} \end{cases}.$$

Each index has n possible choices, there are $r + s$ indexes to choose, so there are n^{r+s} such maps. Moreover, as each map maps some element to zero that all other maps map to one, these maps are all linearly independent. We see from the above calculation that
$\phi = \sum_{(i_1,\ldots,i_r,j_1,\ldots,j_s)\in\{1,\ldots,n\}^{r+s}} \alpha_{1_{i_1}} \ldots \alpha_{r_{i_r}} \beta_{1_{j_1}} \ldots \beta_{s_{j_s}} T_{i_1,\ldots,i_r,j_1,\ldots,j_s}$. Thus these maps form a basis of $\mathcal{T}_r^s(V)$ and so $\dim(\mathcal{T}_r^s(V)) = n^{r+s}$. □

Define the symmetriser $\zeta : \mathcal{T}^r(V) \to \mathcal{T}^r(V)$ by

$$(\zeta\phi)(v_1, \ldots, v_r) = \frac{1}{r!} \sum_{\sigma \in S_r} \phi(v_{\sigma(1)}, \ldots, v_{\sigma(r)})$$

.

Propostion. .

- $\zeta^2 = \zeta$
- $\phi \in \Sigma^r(V) \iff \zeta\phi = \phi$
- $\zeta(\mathcal{T}^r(V)) = \Sigma^r(V)$

Proof. .

- Let $\phi \in \mathcal{T}^r(V), v_1, \ldots, v_r \in V$, then

$$\begin{aligned}
\zeta^2(\phi)(v_1, \ldots, v_r) &= (\zeta(\zeta(\phi)))(v_1, \ldots, v_r) \\
&= \frac{1}{r!} \sum_{\mu \in S_r} (\zeta\phi)(v_{\mu(1)}, \ldots, v_{\mu(r)}) \\
&= \frac{1}{r!} \sum_{\mu \in S_r} \frac{1}{r!} \sum_{\sigma \in S_r} \phi(v_{(\sigma\circ\mu)(1)}, \ldots, v_{(\sigma\circ\mu)(r)}) \\
&= \frac{1}{r!^2} \sum_{\mu \in S_r} \sum_{\sigma \in S_r} \phi(v_{(\sigma\circ\mu)(1)}, \ldots, v_{(\sigma\circ\mu)(r)})
\end{aligned}$$

  Now, composing any permutation $\sigma \in S_r$ with all permutations $\mu \in S_r$ gives all permutations of S_r exactly once. As $|S_r| = r!$, this double sum simply adds up every permutation of S_r $r!$ times, thus we can consolidate the two sums into one as follows.

$$\begin{aligned}
&= \frac{1}{r!} \sum_{\alpha \in S_r} \phi(v_{\alpha(1)}, \ldots, v_{\alpha(r)}) \\
&= (\zeta\phi)(v_1, \ldots, v_r)
\end{aligned}$$

  Thus $\zeta^2 = \zeta$.

-
  - suppose $\zeta\phi = \phi$, then $\frac{1}{r!} \sum_{\sigma \in S_r} \phi(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) = \phi(v_1, \ldots, v_r)$. The for any $\mu \in S_r$,

$$\phi(v_{\mu(1)}, \ldots, v_{\mu(r)}) = \frac{1}{r!} \sum_{\sigma \in S_r} \phi(v_{(\sigma\circ\mu)(1)}, \ldots, v_{(\sigma\circ\mu)(r)}) = (\zeta\phi)(v_1, \ldots, v_r) = \phi(v_1, \ldots, v_r)$$

    Thus $\phi \in \Sigma^r(V)$.
  - Suppose $\phi \in \Sigma^r(V)$, then

$$\begin{aligned}
(\zeta\phi)(v_1, \ldots, v_n) &= \frac{1}{n!} \sum_{\sigma \in S_r} \phi(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) \\
&= \frac{1}{r!} \sum_{\sigma \in S_r} \phi(v_1, \ldots, v_r) \\
&= \frac{1}{r!} r! \phi(v_1, \ldots, v_r) \\
&= \phi(v_1, \ldots, v_r)
\end{aligned}$$

    Thus $\zeta\phi = \phi$

  Hence $\phi \in \Sigma^r(V) \iff \zeta\phi = \phi$.

- It is clear that $\zeta\phi \in \Sigma^r(V)$ for any $\phi \in \mathcal{T}^r(V)$, and so we have $\zeta(\mathcal{T}^r(V)) \subseteq \Sigma^r(V)$. Moreover, for any $\phi \in \Sigma^r(V)$, by the above $\phi = \zeta\phi$, and so $\phi \in \zeta(\mathcal{T}^r(V))$. Thus $\Sigma^r(V) \subseteq \zeta(\mathcal{T}^r(V))$, and so $\zeta(\mathcal{T}^r(V)) = \Sigma^r(V)$.

□

Define the wedge product $\cdot \wedge \cdot : \mathcal{T}^r(V) \times \mathcal{T}^s(V) \to \Lambda^{r+s}(V)$ by $\phi \wedge \psi = \frac{(r+s)!}{r!s!}\mathcal{A}(\phi \otimes \psi)$, where $\mathcal{A} : \mathcal{T}^r(V) \to \mathcal{T}^r(V)$ is defined by $\mathcal{A}(\phi)(v_1, \ldots, v_r) = \frac{1}{r!}\sum_{\sigma \in S_r} \operatorname{sgn}(\sigma)\phi(v_{\sigma(1)}, \ldots, v_{\sigma(r)})$.

Propostion. The wedge product is bilinear

Proof. Let $\phi_1, \phi_2 \in \mathcal{T}^r(V), \psi \in \mathcal{T}^s(V), a_1, a_2 \in \mathbb{R}$, and let $K = \frac{(r+s)!}{r!s!}$. Then

$$
\begin{aligned}
(a_1\phi_1 + a_2\phi_2) \wedge \psi &= \frac{(r+s)!}{r!s!}\mathcal{A}((a_1\phi_1 + a_2\phi_2) \otimes \psi) = \\
&= K\frac{1}{(r+s)!}\sum_{\sigma \in S_{r+s}} \operatorname{sgn}(\sigma)(a_1\phi_1 + a_2\phi_2)(v_{\sigma(1)}, \ldots, v_{\sigma(r)})\psi(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)}) \\
&= K\frac{1}{(r+s)!}\sum_{\sigma \in S_{r+s}} \operatorname{sgn}(\sigma)[a_1\phi_1(v_{\sigma(1)}, \ldots, v_{\sigma(r)})\psi(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)}) \\
&\quad + a_2\phi_2(v_{\sigma(1)}, \ldots, v_{\sigma(r)})\psi(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)})] \\
&= K\frac{a_1}{(r+s)!}\sum_{\sigma \in S_{r+s}} \operatorname{sgn}(\sigma)(\phi_1 \otimes \psi)(v_{\sigma(1)}, \ldots, v_{\sigma(r+s)}) \\
&\quad + K\frac{a_2}{(r+s)!}\sum_{\sigma \in S_{r+s}} \operatorname{sgn}(\sigma)(\phi_2 \otimes \psi)(v_{\sigma(1)}, \ldots, v_{\sigma(r+s)}) \\
&= a_1(\phi_1 \wedge \psi) + a_2(\phi_2 \wedge \psi)
\end{aligned}
$$

The proof for linearity in the second argument is identical. Thus the wedge product is bilinear. □

Propostion. Recalling that $n = \dim(V)$,

- if $r > n$, then $(\Lambda^r(V)) = 0$
- for $0 \leq r \leq n, \dim(\Lambda^r(V)) = \binom{n}{r}$
- $\dim(\Lambda(V)) = 2^n$

Proof. Consider $r > n$ and let $\phi \in \mathcal{T}^r(V)$ be alternating, $v_1, \ldots, v_r \in V$. As $r > n$, the v_i are linearly dependent, so there exist $\alpha_1, \ldots, \alpha_{r-1}$ such that $v_r = \alpha_1 v_1 + \ldots + \alpha_{r-1}v_{r-1}$. Thus

$$\phi(v_1, \ldots, v_r) = \alpha_1\phi(v_1, \ldots, v_{r-1}, v_1) + \ldots + \alpha_{r-1}\phi(v_1, \ldots, v_{r-1}, v_{r-1})$$

Now, each of the outputs on the right hand side has a repeated argument, and so is zero as ϕ is alternating. Thus $\phi(v_1, \dots, v_r) = 0$, and so $\phi = 0$. Clearly 0 is alternating and so $\Lambda^r(V) = 0$.

Now consider $0 \le r \le n$, and let $\phi \in \Lambda^r(V)$. As ϕ is a multilinear map, we need only define ϕ on all the basis elements $(e_{i_1}, \dots, e_{i_r})$ of V^r. Moreover, as ϕ is alternating, defining it on any basis element also determines it on all basis elements formed by re-arranging the indexes (e.g. taking for example $n = 3$, defining ϕ on (e_1, e_2, e_3) also defines it one (e_2, e_3, e_1)). Defining the equivalence relation on basis elements of V^r

$$(e_{i_1}, \dots, e_{i_r}) \sim (e_{j_1}, \dots, e_{j_r}) \iff (i_1, \dots, i_r) \text{ and } (j_1, \dots, j_r) \text{ are permutations of each other}$$

we see that to uniquely define ϕ we need only define ϕ on all the equivalence classes of this relation.

Now, each equivalence class is uniquely defined by picking r indexes from $\{1, \dots, n\}$. Thus there are $\binom{n}{r}$ equivalence classes, and thus $\dim(\Lambda^r(V)) = \binom{n}{r}$.

Finally, the above facts and the binomial theorem gives that

$$\begin{aligned}
\dim(\Lambda(V)) &= \sum_{i=0}^{\infty} \dim(\Lambda^i(V)) \\
&= \sum_{i=0}^{n} \binom{n}{i} + \sum_{i=n+1}^{\infty} 0 \\
&= \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^i \\
&= (1+1)^n \\
&= 2^n
\end{aligned}$$

□