Propostion. Let $\mathcal{M}, \mathcal{N}$ be smooth manifolds and let ϕ be a Riemannian metric on $\mathcal{N}$. If $F : \mathcal{M} \to \mathcal{N}$ is an immersion, then $F_*\phi$ is a Riemannian metric on $\mathcal{M}$.

Proof. We must show that $F_*\phi$ has the required properties. Let $p \in \mathcal{M}$, $X_p, Y_p \in \mathcal{T}_p(\mathcal{M})$, then

- **Symmetry**:

$$\begin{aligned}(F_*\phi)_p(X_p, Y_p) &= \phi_p(F_{*_p}X_p, F_{*_p}Y_p) \\ &= \phi_p(F_{*_p}Y_p, F_{*_p}X_p) \qquad \text{by symmetry of } \phi_p \\ &= (F_*\phi)_p(Y_p, X_p)\end{aligned}$$

  Thus $(F_*\phi)_p$ is symmetric for all $p \in \mathcal{M}$.

- **Positive Definiteness**:

$$\begin{aligned}(F_*\phi)_p(X_p, X_p) &= \phi_p(F_{*_p}X_p, F_{*_p}X_p) \\ &\geq 0 \qquad \text{by positive definiteness of } \phi_p\end{aligned}$$

  So $(F_*\phi)_p$ is positive definite for all $p \in \mathcal{M}$.

- **Seperation of Points**:

$$\begin{aligned}(F_*\phi)_p(X_p, X_p) = 0 &\iff \phi_p(F_{*_p}X_p, F_{*_p}X_p) = 0 \\ &\iff F_{*_p}X_p = 0 \qquad \text{as } \phi \text{ is a Riemannian metric} \\ &\iff X_p = 0 \qquad \text{as } F \text{ is an immersion, and so } F_{*_p} \text{ is injective}\end{aligned}$$

Thus $F_*\phi$ is a Riemannian metric on $\mathcal{M}$. □