Propostion. F_p^* is a ring homomorphism

Proof. Let $f, g \in C^\infty(F(p))$, then for any $x \in \mathcal{M}$ st $f(x), g(x)$ are defined. Then

- $$\begin{aligned} F_p^*(f+g)(x) &= (f+g)(F(x)) \\ &= f(F(x)) + g(F(x)) \\ &= F_p^*(f)(x) + F_p^*(g)(x) \\ &= (F_p^*(f) + F_p^*(g))(x) \end{aligned}$$

  hence $F_p^*(f+g) = F_p^*(f) + F_p^*(g)$

- $$\begin{aligned} F_p^*(fg)(x) &= (fg)(F(x)) \\ &= f(F(x))g(F(x)) \\ &= F_p^*(f)(x)F_p^*(g)(x) \\ &= (F_p^*(f)F_p^*(g))(x) \end{aligned}$$

  hence $F_p^*(fg) = F_p^*(f)F_p^*(g)$

Now consider the identity function $1_{C^\infty(F(p))}$ (more formally, the equivalence class of functions that are identically equal to 1 on some open neighbourhood of $F(p)$, then $F_p^*(1_{C^\infty(F(p))})(x) = 1_{C^\infty(F(p))}(F(x)) = 1$, and so $F(1_{C^\infty(F(p))}) = 1_{C^\infty(p)}$. Thus F_p^* is a ring homomorphism. □

Propostion. F_{*_p} is a linear map.

Proof. Let $X_p, Y_p \in T_p(\mathcal{M}), \alpha, \beta \in \mathbb{R}$, then for any $f \in C^\infty(p)$

$$\begin{aligned} F_{*_p}(\alpha X_p + \beta Y_p)(f) &= (\alpha X_p + \beta Y_p)(f \circ F) \\ &= \alpha X_p(f \circ F) + \beta Y_p(f \circ F) \\ &= \alpha F_{*_p}(X_p)(f) + \beta F_{*_p}(Y_p)(f) \\ &= (\alpha F_{*_p}(X_p) + \beta F_{*_p}(Y_p))(f) \end{aligned}$$

Thus $F_{*_p}(\alpha X_P + \beta Y_p) = \alpha F_{*_p}(X_p) + \beta F_{*_p}(Y_p)$, and so F_{*_p} is a linear map. □