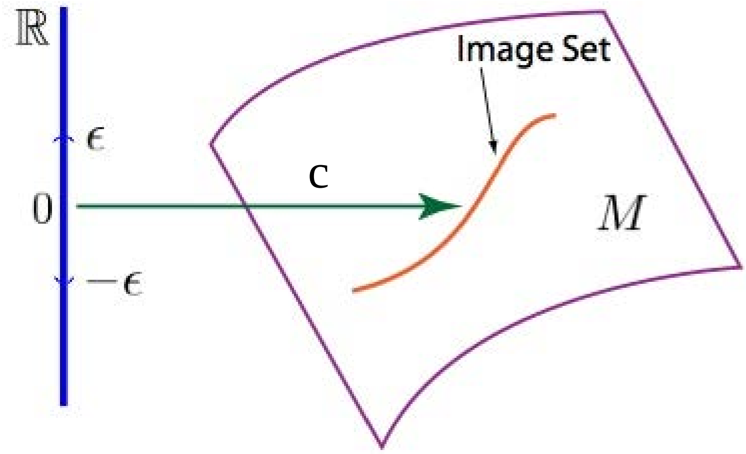
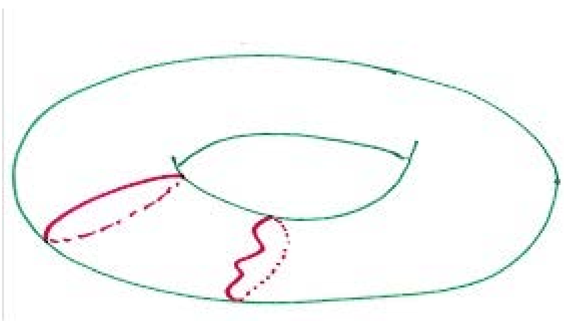
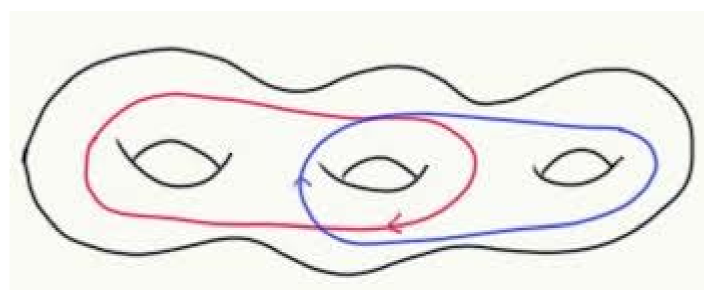
Curve on a manifold

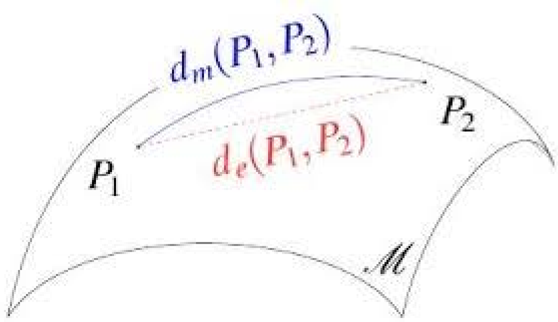




Homotopic curves

Non homotopic curves





Metric on a manifold, and

Euclidean metric

Curve in $\mathbb{R}^3$: The Frenet frame

Consider a parametric curve in $\mathbb{R}^3$, it means a map $c : [a,b] \to \mathbb{R}^3$, $t \mapsto c(t)$, and we assume it smooth.

This curve is regular if $\dot{c}(t) = \frac{d}{dt} c(t) \neq 0$.

The arc length is defined by $s = s(t) = \int_a^t \|\dot{c}(t)\| \, dt$ with $\|\dot{c}(t)\|$ the Euclidean norm of $\dot{c}(t)$ in $\mathbb{R}^3$.

Let $L := \int_a^b \|\dot{c}(t)\| \, dt$ the length of the curve.

Lemma: If c is regular, $\exists$ a diffeomorphism $\phi : [0,L] \to [a,b]$ such that $\|(c\circ\phi)'(s)\| = 1$ $\forall s \in [0,L)$.

We say that the curve is parameterized by its arc length, and in this parametrization the tangent vector is of length 1.

see any course of calculus II for the proof, or [Klingenberg] p 9.

Whenever the letter s is used for the parametrization of a curve, it means that it is the arc length parametrization.

Let us set $T(s) := (c \circ \phi)'(s)$, and observe that

constant function $\qquad$ scalar product in $\mathbb{R}^3$

$$0 = \frac{d}{ds} 1 = \frac{d}{ds} \| T(s) \|^2 = \frac{d}{ds} \langle T(s), T(s) \rangle =$$

$$= \langle \dot{T}(s), T(s) \rangle + \langle T(s), \dot{T}(s) \rangle = 2 \langle T(s), \dot{T}(s) \rangle$$

symmetry of the scalar product in $\mathbb{R}^3$

with $\dot{T}(s) := \frac{d}{ds} T(s)$.

Thus $\underline{\dot{T}(s) \perp T(s)}$ ($\dot{T}(s)$ is perpendicular to $T(s)$ since their scalar product is 0).

Set $K(s) := \| \dot{T}(s) \|$ and call it the <u>curvature</u>.

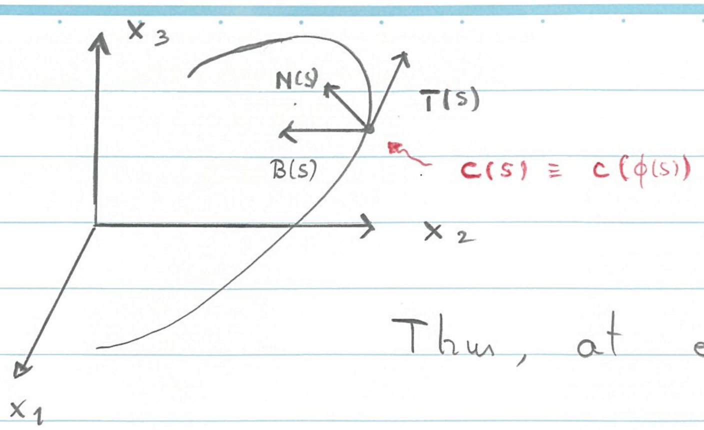
If $K(s) \neq 0$, we set $N(s)$ for the vector of norm 1 satisfying

positive scalar

$$\dot{T}(s) = K(s)\, N(s)$$

vector in $\mathbb{R}^3$ $\qquad$ vector in $\mathbb{R}^3$

If $K(s) \neq 0$ we also set $B(s) \in \mathbb{R}^3$ for the unique vector of norm 1 such that $\{ T(s), N(s), B(s) \}$ is a basis of $\mathbb{R}^3$ (with a positive orientation).



Thus, at every point of the curve where the curvature $\kappa(s) \neq 0$, one can define an orthonormal basis, it corresponds to a field of orthonormal frames.

<u>Observation</u>: One can show that $\kappa(s) = 0 \ \forall s \in I \iff$ the curve is a straight line on this interval I.

Let us set $F_1(s) := T(s)$, $F_2(s) = N(s)$, $F_3(s) := B(s)$ (we assume $\kappa(s) \neq 0$). Since these vectors generate an orthonormal basis, one has $\langle F_i(s), F_j(s) \rangle = \delta_{ij}$ and $\frac{d}{ds} \langle F_i(s), F_j(s) \rangle = \langle \dot{F}_i(s), F_j(s) \rangle + \langle F_i(s), \dot{F}_j(s) \rangle = 0$ ⊛

Since $\dot{F}_i(s)$ is a linear combination of the 3 vectors $F_1(s), F_2(s)$, $F_3(s)$ one has $\dot{F}_j(s) = \sum_{k=1}^{3} \underset{\substack{\cap \\ \mathbb{R}}}{a_j^k} F_k(s)$ for $j = 1, 2, 3$.

$\| \ a_j^k(s)$

By inserting this in ⊛ one gets

$$\langle \sum_k \dot{a}_i^k F_k(s), F_j(s) \rangle + \langle F_i(s), \sum_k \dot{a}_j^k F_k(s) \rangle = 0$$

$\Leftrightarrow \quad a_i^j(s) + a_j^i(s) = 0 \quad \Rightarrow \quad (a_i^j(s))_{i,j}$ is a

skew-symmetric matrix (in particular $a_i^i(s) = 0$)

Also, since $\dot{F}_1(s) = \dot{T}(s) = K(s)\,N(s) = K(s)\,F_2(s) \Rightarrow a_1^2(s) = K(s)$

and $a_1^3(s) = 0$. Let us finally set $a_2^3(s) =: \mathcal{T}(s)$

and call it the <u>torsion</u>. One finally gets the system

$$\begin{cases} \dot{T}(s) = K(s)\,N(s) \\ \dot{N}(s) = -K(s)\,T(s) + \mathcal{T}(s)\,B(s) \\ \dot{B}(s) = -\mathcal{T}(s)\,N(s) \end{cases}$$

Frenet-Serret formulas

This system determines the evolution of the tangent vector N(s), the normal vector N(s) and the binormal vector B(s) along the curve c.

<u>Lemma</u>: The curve lies in a plane iff $\mathcal{T}(s) = 0 \ \forall s$.

[Bo, Thm 1.9 p 303]