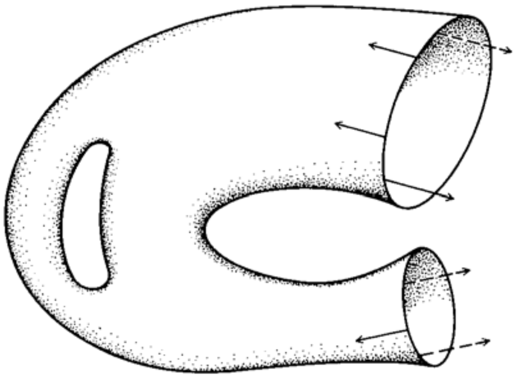
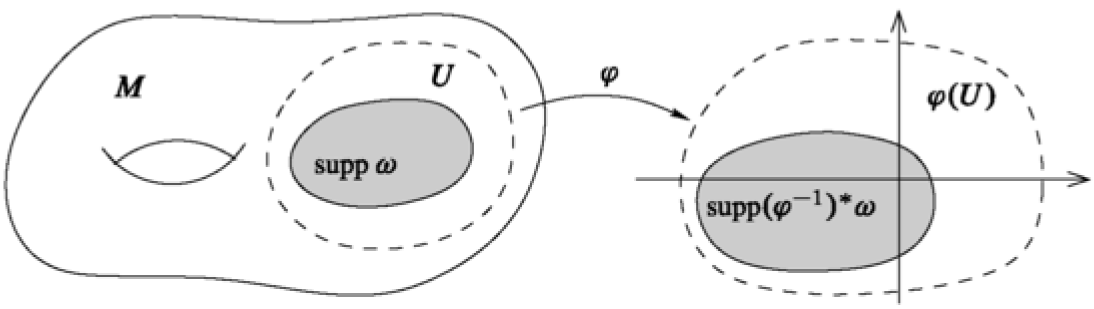
Appendix VIII



Manifold with boundary with inward and outward pointing vectors at the boundary

Differential form ω with compact support, and its local representation

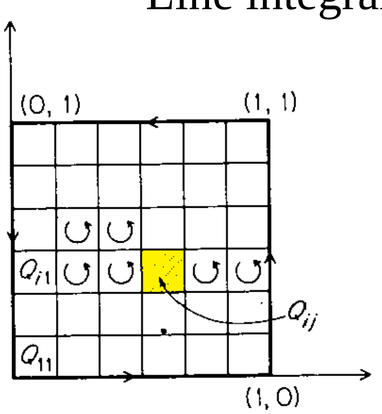


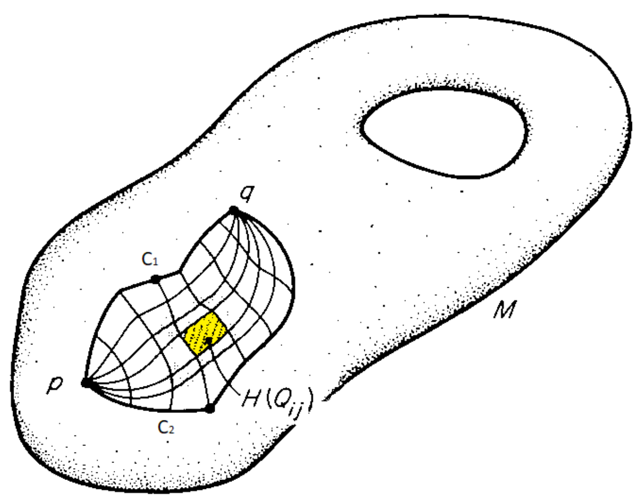
Stokes's theorem

Theorem. *(Stokes–Cartan) If ω is a smooth $(n-1)$-form with compact support on smooth n-dimensional manifold-with-boundary Ω, $\partial\Omega$ denotes the boundary of Ω given the induced orientation, and $i : \partial\Omega \hookrightarrow \Omega$ is the inclusion map, then*

$$\int_\Omega d\omega = \int_{\partial\Omega} i^*\omega.$$

Line integrals





and independence with respect to the path if dω = 0

Remark on Stokes's theorem, from [Bo] p 263.

It is important to note that the version of Stokes's theorem proved above is deficient in the following sense: it holds only for smooth manifolds with smooth boundary. Thus, for example, our proof does not even include the case of a square in $\boldsymbol{R}^2$ or an open set of $\boldsymbol{R}^3$ bounded by a polyhedron. The difficulty in these cases is not so much with the analysis and integration theory, as with describing the regions of integration to be admitted and with giving precise definitions of orientability and induced orientation of the boundary. The search for reasonable domains of integration to validate Stokes's theorem usually leads to the concept of a simplicial or polyhedral complex, that is, a space made up by fastening together along their faces a number of simplices (line segments, triangles, tetrahedra, and their generalizations) (Fig. VI.4) or more general polyhedra (cubes, for example). Since it can be shown (see Munkres [1]) that any C^∞ manifold M may be "triangulated," which means that it is homeomorphic (even with considerable smoothness) to such a complex, the integral over M becomes the sum of the integrals over the pieces, which are images of simplices, cubes, or other polyhedra as the case may be (compare Remark 2.7). The strategy is then to reduce the theory (including Stokes's theorem) to the case of polyhedral domains of $\boldsymbol{R}^n$. This approach is particularly important for those interested in algebraic topology and de Rham's theorem. It is very clearly set forth, for example, by Singer and Thorpe [1] or Warner [1].

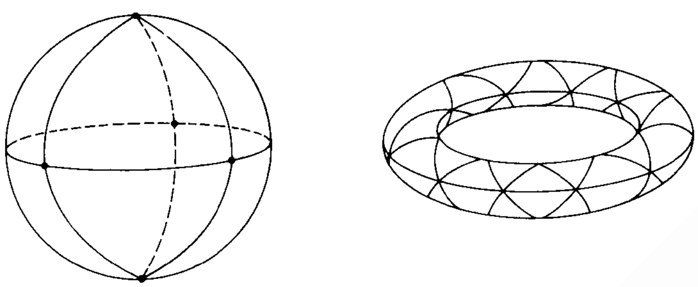
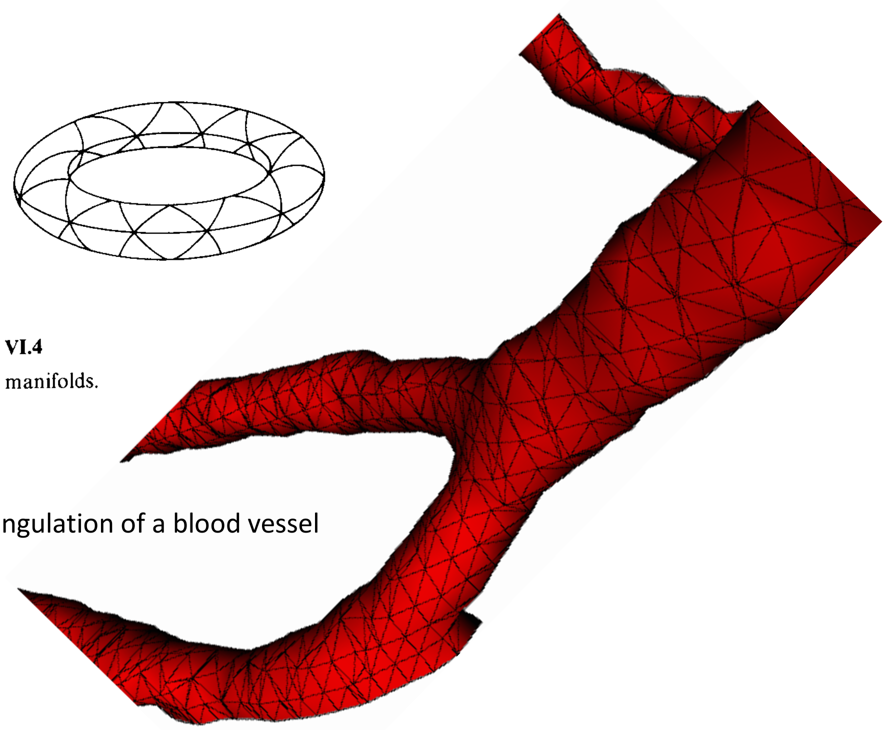


Figure VI.4
Triangulated manifolds.



Triangulation of a blood vessel