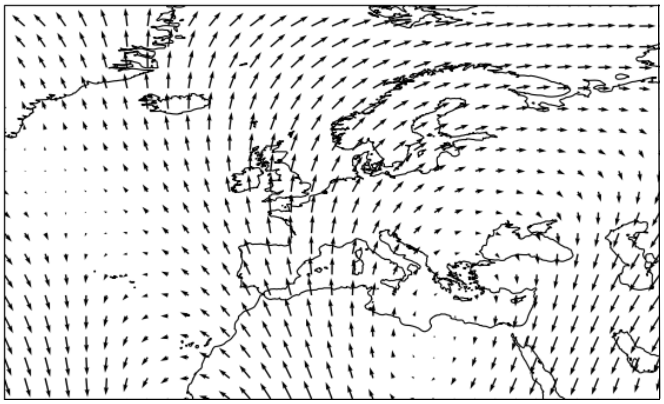
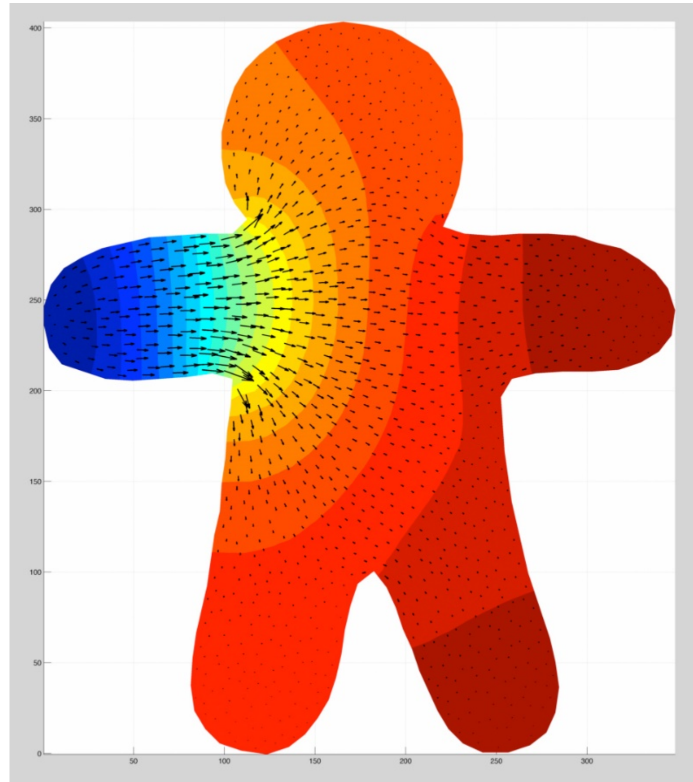
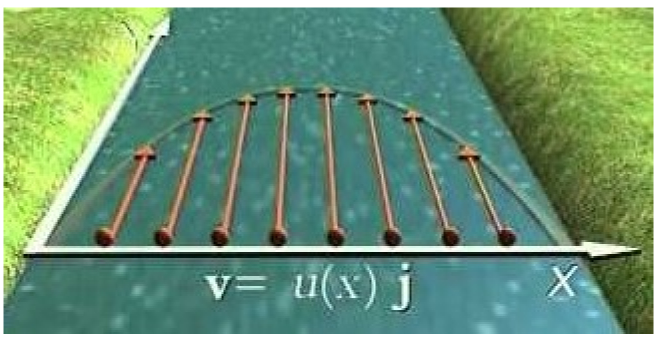
Appendix VI

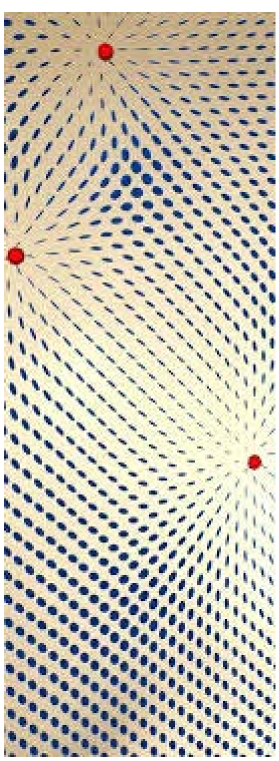
Vector fields

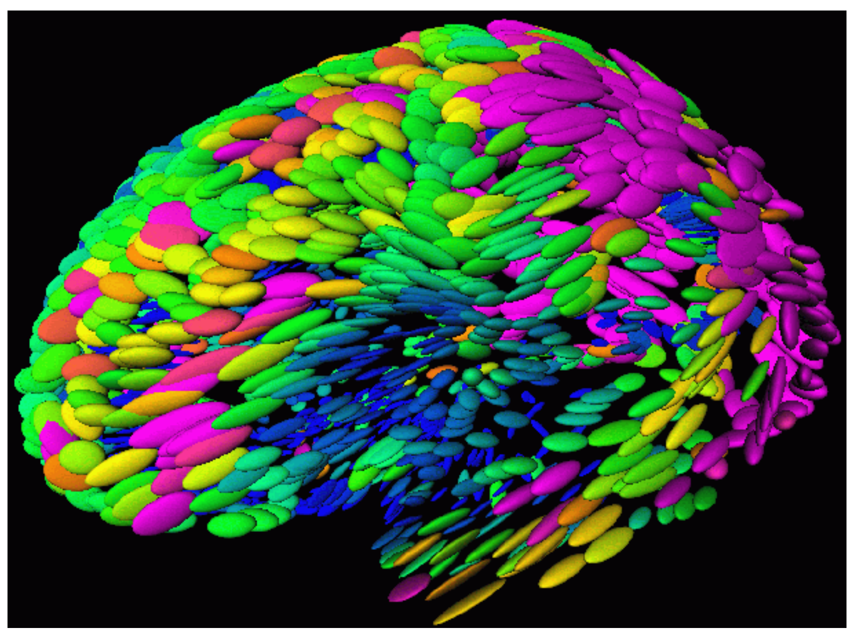






Tensor fields





Striking differences are found, even among normal human subjects, in the gyral patterns of the cerebral cortex. **Tensor maps** can be used to visualize these complex patterns of anatomical variation. In these maps (*below*), color distinguishes regions of high variability (*pink colors*) from areas of low variability (*blue*). Rectangular glyphs indicate the principal directions of variation - they are most elongated along directions where there is greatest anatomic variation across subjects. Each glyph represents the covariance tensor of the vector fields that map individual subjects onto their group average anatomic representation. The maps are based on a group of 40 normal subjects. The resulting information can be leveraged to distinguish normal from abnormal anatomical variants using random tensor field algorithms.

http://users.loni.usc.edu/~thompson/TMP/tensor.html

Reminder from Calculus II

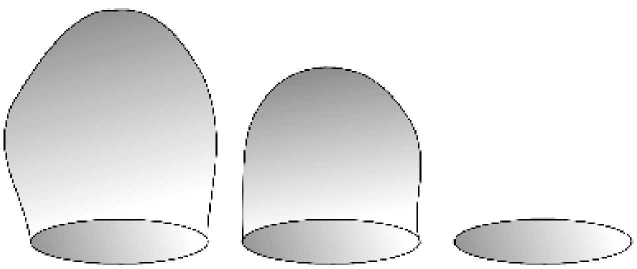
Stokes' Theorem:

$$\boxed{\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint (\underbrace{\boldsymbol{\nabla} \times \vec{\mathbf{F}}}_{\text{curl}\vec{\mathbf{F}}}) \cdot \hat{\mathbf{n}}\, dS}$$

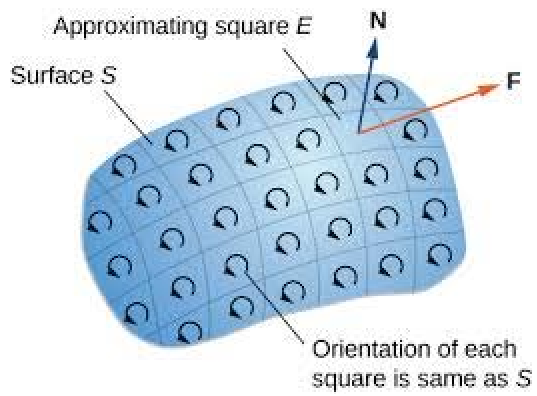
if C closed curve

S = any surface bounded by C

and $\vec{\mathbf{F}}$ defined everywhere in S



Different surfaces, same boundary



Integration

- Line*

$$\int_\gamma \vec{F} \cdot d\vec{r}$$

- Surface*

$$\iint_S \vec{F} \cdot d\vec{S}$$

- Volume

$$\iiint_V \rho\, dV$$

Differentiation

- grad

$$\vec{F} = -\vec{\nabla} \cdot \phi$$

- curl

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

- div

$$\rho = \vec{\nabla} \cdot \vec{E}$$

Both

- Gradient Theorem:

$$\int_{\gamma \text{ from } \vec{a} \to \vec{b}} \left(-\vec{\nabla}\phi\right) \cdot d\vec{r} = \phi(\vec{a}) - \phi(\vec{b})$$

independent of path γ

- Stoke's Theorem:

$$\iint_S \left(\vec{\nabla} \times \vec{A}\right) \cdot d\vec{S} = \oint_{\partial S} \vec{A} \cdot d\vec{r}$$

- Divergence Theorem:

$$\iiint_V \left(\vec{\nabla} \cdot \vec{E}\right) dV = \oiint_{\partial V} \vec{E} \cdot d\vec{S}$$

* Also for scalar fields: $\int_\gamma \lambda\, |d\vec{r}|$ and $\iint_S \sigma\, \left|d\vec{S}\right|$

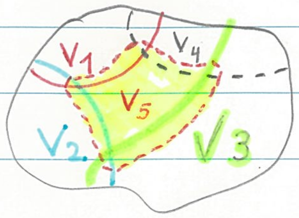
Important: It does not matter how I call my fields (F, E, A, ...). The theorems apply independent of the letter I use.

All summarized in: $$\int_{\partial\Omega} \omega = \int_\Omega d\omega\,.$$

Partition of unity

Recall that a cover of $\mathcal{M}$ is a collection $\{V_\alpha\}_\alpha$ of subsets of $\mathcal{M}$ such that $\bigcup_\alpha V_\alpha = \mathcal{M}$.

union of all V_α



The cover is locally finite if $\forall p \in \mathcal{M}$, there exits a neighborhood U of p such that $U \cap V_\alpha = \emptyset$ except for a finite number of α.

The intersection of U with most of the V_α is empty.

Recall also that if $f : \mathcal{M} \to \mathbb{R}$, its support is defined by $\mathrm{supp}(f) := \overline{\{ p \in \mathcal{M} \mid f(p) \neq 0 \}}$

we take the closure of this set $\{\dots\}$, it means we consider the smallest closed set containing $\{\dots\}$.

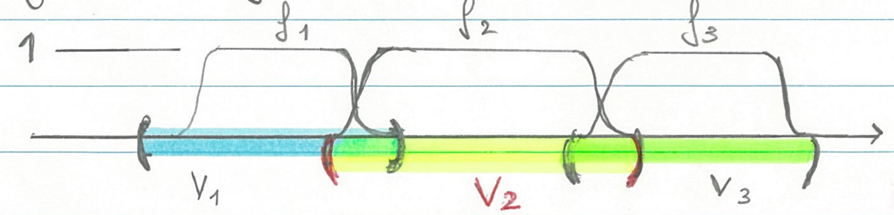
The following definition has many aplications:

Def : A <u>smooth partition of unity of $\mathcal{M}$</u> is a collection $\{f_\beta\}_\beta$ of smooth functions on $\mathcal{M}$ satisfying :

1) $f_\beta \geqslant 0$ the functions are positive
2) $\{\mathrm{supp}(f_\beta)\}_\beta$ is a locally finite cover of $\mathcal{M}$,
3) $\sum_\beta f_\beta(p) = 1 \quad \forall p \in \mathcal{M}$.

for every p this sum is finite because of 2)

This partition is <u>subordinate</u> to an open cover $\{V_\alpha\}_\alpha$ of $\mathcal{M}$ if $\forall \beta \; \exists V_\alpha$ with $\mathrm{supp}(f_\beta) \subset V_\alpha$.



Thm : For any smooth manifold $\mathcal{M}$ and for any open cover of $\mathcal{M}$ there exists a smooth partition of unity of $\mathcal{M}$ subordinated to the open cover.