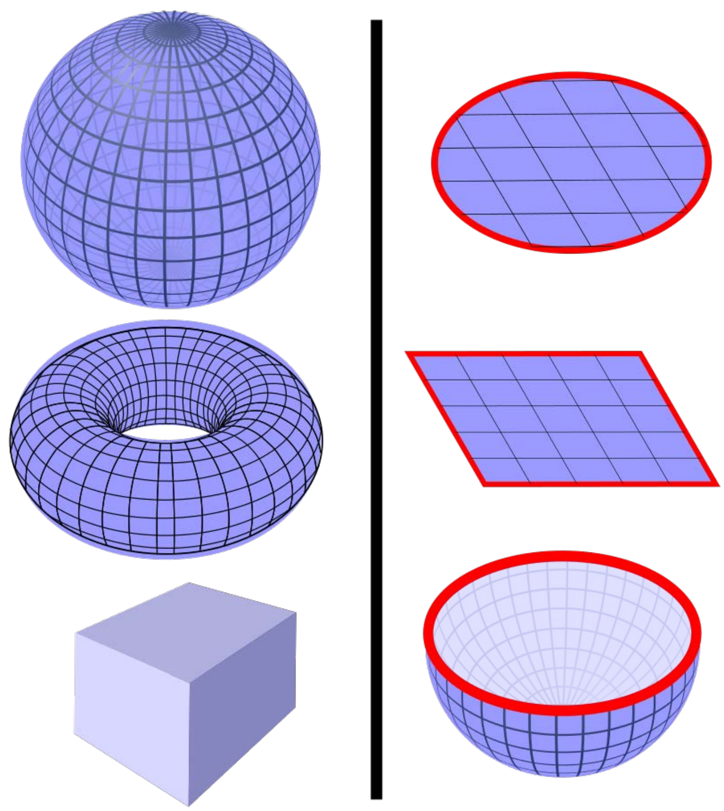
Appendix II

Homeomorphic manifolds

Topological manifolds without or with boundary



Differential structures on spheres of dimension 1 to 20 [edit]

The following table lists the number of smooth types of the topological m−sphere $\mathbf{S}^m$ for the values of the dimension m from 1 up to 20. Spheres with a smooth, i.e. C^∞−differential structure not smoothly diffeomorphic to the usual one are known as exotic spheres.

Dimension	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Smooth types	1	1	1	≥1	1	1	28	2	8	6	992	1	3	2	16256	2	16	16	523264	24

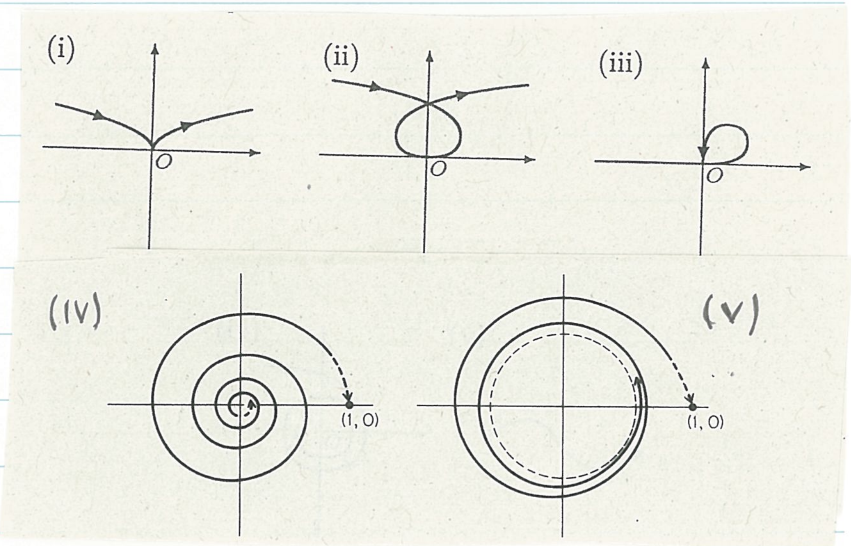
Immersion, submersion, submanifold

⚠ These definitions are not universal and can be slightly different depending on the authors.

<u>Def</u>: Let $f : \mathcal{M} \to \mathcal{N}$ be a smooth map between smooth manifolds of dim m and n respectively.

- f is an <u>immersion</u> if $\operatorname{rank}(f)_p = m$ for any $p \in \mathcal{M}$.
- f is a <u>submersion</u> if $\operatorname{rank}(f)_p = n$ for any $p \in \mathcal{M}$.

<u>Examples</u>: $\mathcal{M} = \mathbb{R}$, $\mathcal{N} = \mathbb{R}^2$



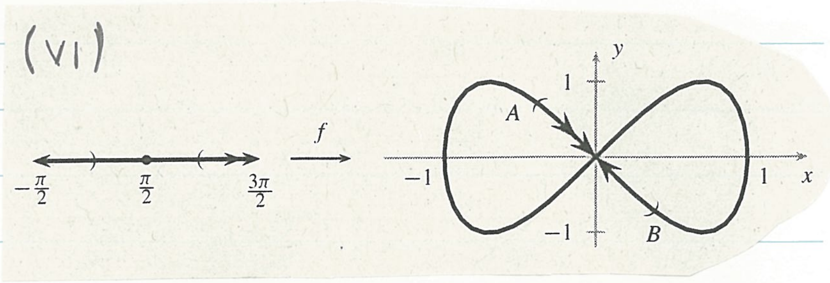
i) $f : \mathbb{R} \ni t \mapsto (t^3, t^2) \in \mathbb{R}^2$ is not an immersion since $\left[\frac{d}{dt} f\right](0) = (0,0)$.

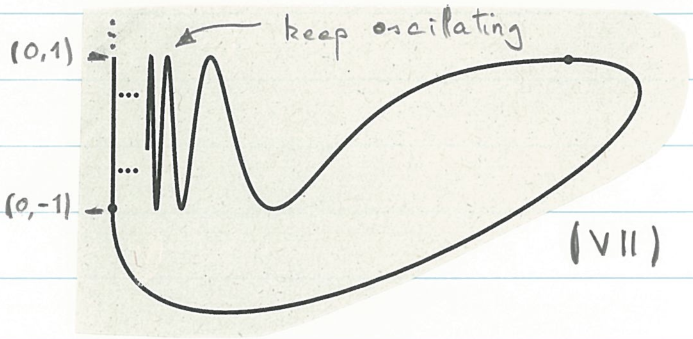
All the other examples are immersion since $\left[\frac{d}{dt} f\right](t) = f'(t) \neq (0,0)$ $\forall t \in \mathbb{R}$.

<u>Remarks</u>: 1) An immersion can be injective, or not (like in (ii)).

2) If f is an injective immersion, then $f(\mathcal{M})$ can be endowed with the topology and the differential structure of $\mathcal{M}$, but this is not really interesting since $\mathcal{N}$ does not play any role. In this situation, $f(\mathcal{M})$ is called <u>a submanifold</u> or <u>immersed submanifold</u>.

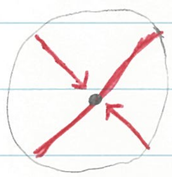
Injective immersions can be of different nature when the topology of N is taken into account, even through the subspace ($\equiv$ relative) topology.





For example, look at $(0,0)$ in (VI). In the relative topology any neighborhood of $(0,0)$ contain 3 parts parts:

neighborhood of $(0,0)$ in the relative topology



neighborhood of $(0,0)$ in the topology of $f(\mathbb{R})$.

The same phenomenon takes place on any point on $(0, x)$ with $x \in (-1, 1)$ in (VII), but the number of disconnected parts is even infinite!

<u>Definition</u>: A smooth map $f: M \to N$ is called an <u>imbedding</u> (or <u>embedding</u>) if it is an injective immersion and f defines a homeomorphism between M and $f(M)$ when $f(M)$ is endowed with the relative topology inherited from N. $f(M)$ is called an <u>imbedded manifold</u>.

immersed submanifold

$\equiv$

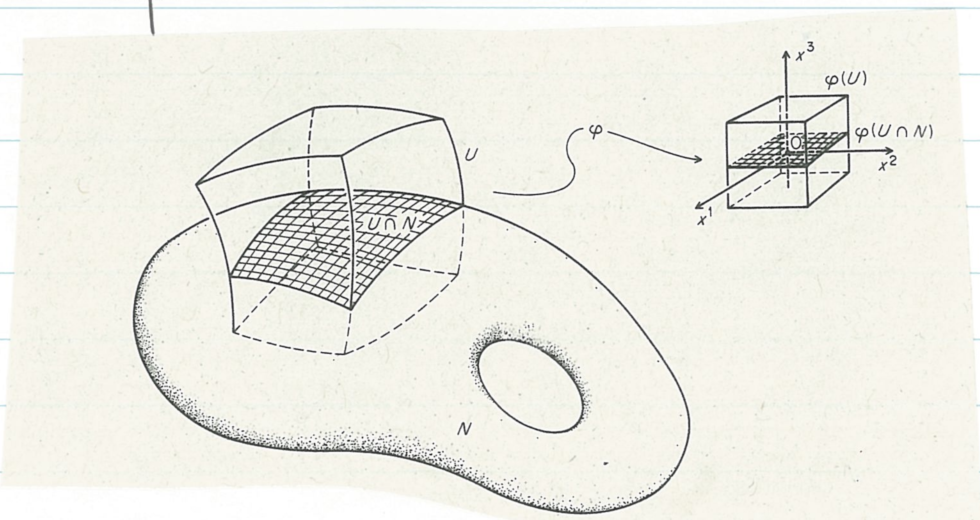
Recall (from p 1) that a submanifold is the image of a manifold $\mathcal{R}$ through a smooth map $f: \mathcal{R} \to N$, when $f(\mathcal{R})$ is endowed with the structure from $\mathcal{R}$. In this case f is a diffeomorphism between $\mathcal{R}$ and $f(\mathcal{R})$.

Def: A subset N of a C^∞-manifold $\mathcal{R}$ has the n-submanifold property if $\forall p \in N$, there exists a chart (U, φ) on $\mathcal{R}$ with $p \in U$ s.t.

1) $\varphi(p) = 0 \in \mathbb{R}^m$
2) $\varphi(U) = C_\varepsilon^m(0)$ ← cube in $\mathbb{R}^m$, centered at 0 and of side 2ε.
3) $\varphi(U \cap N) = \{x \in C_\varepsilon^m(0) \mid x^{n+1} = x^{n+2} = \dots = x^m = 0\}$.

Such a chart is called preferred coordinates or adapted chart with respect to N.

Example with $\mathcal{R} = \mathbb{R}^3$.



Note that an immersed submanifold does not always have this property.

Def : A regular submanifold of a smooth manifold M is a subset N of M with the n-submanifold property and with the C^∞-structure provided by the preferred coordinates charts.

Examples of regular submanifolds are provided in [Tu, p 97].

The first version of the following theorem has been proved by Whitney in 1936. It has then been simplified but it is still called the Whitney imbedding theorem :

Thm : Any smooth manifold M of dimension n can be imbedded in $\mathbb{R}^{2n}$. The image in $\mathbb{R}^{2n}$ is closed.

extrinsic and intrinsic approach are related, but this imbedding is not always so useful.