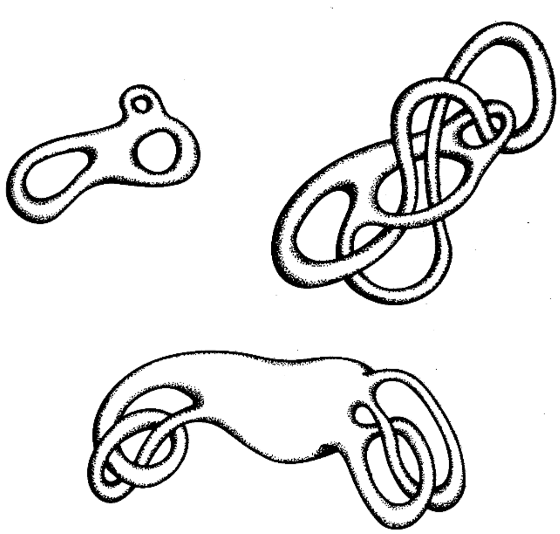
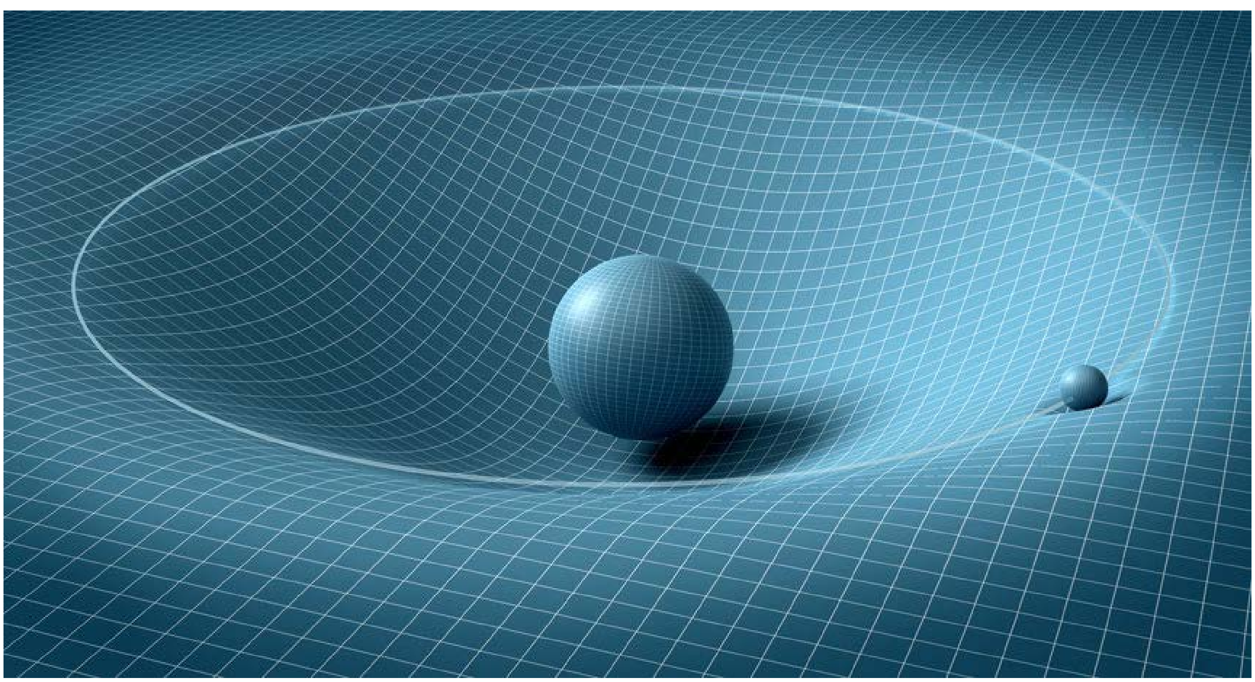
Special Mathematics Lecture: Differential geometry

Extrinsic / **intrinsic**

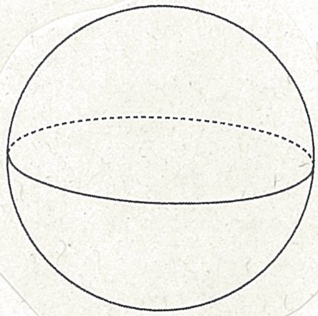




Example of topological manifolds

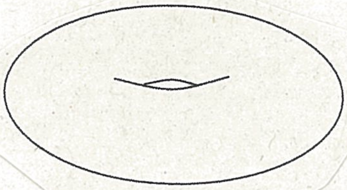
Circle

$$S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1 \}$$



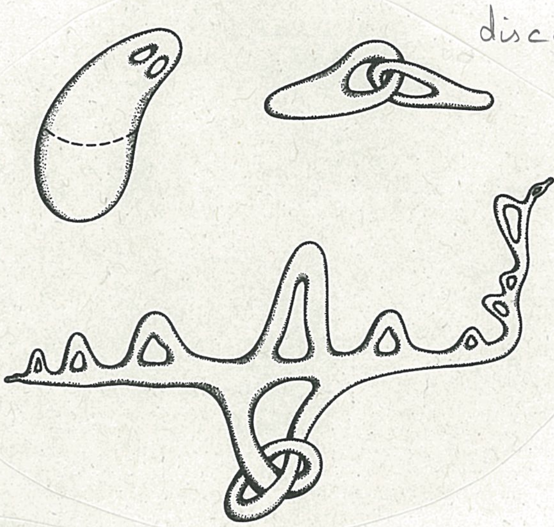
2 - sphere

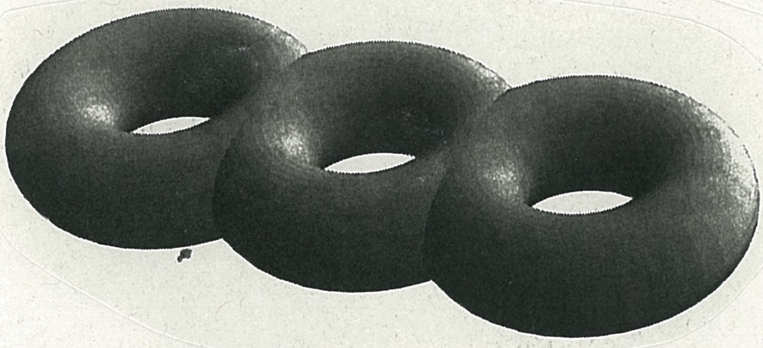
$$S^2 = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1 \}$$



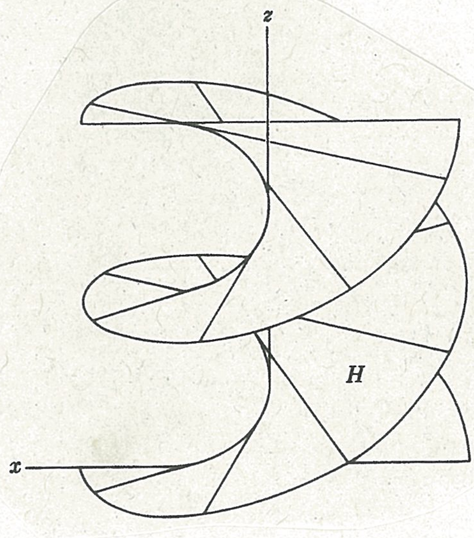
2 - torus

disconnected





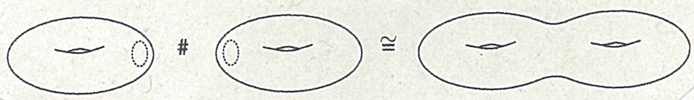
Hyperboloid (in 1 sheet)



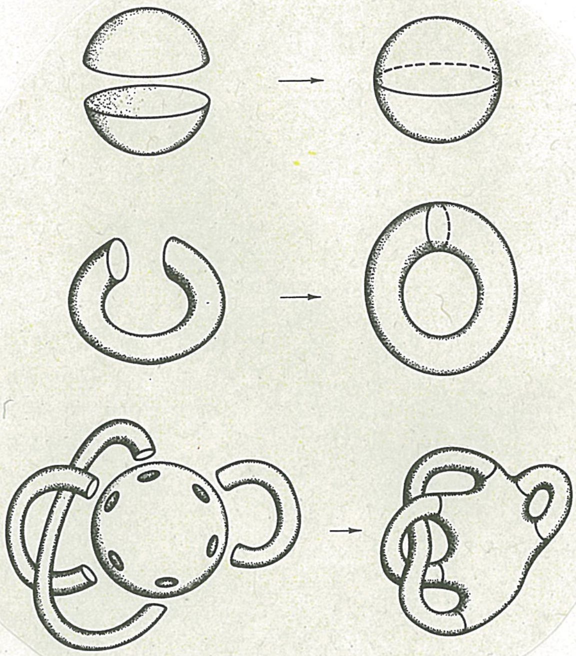
$$\begin{cases} x = s\cos(t) \\ y = s\sin(t) \\ z = bt \end{cases}$$

for fixed $b \neq 0$ and $(s,t) \in \mathbb{R}^2$

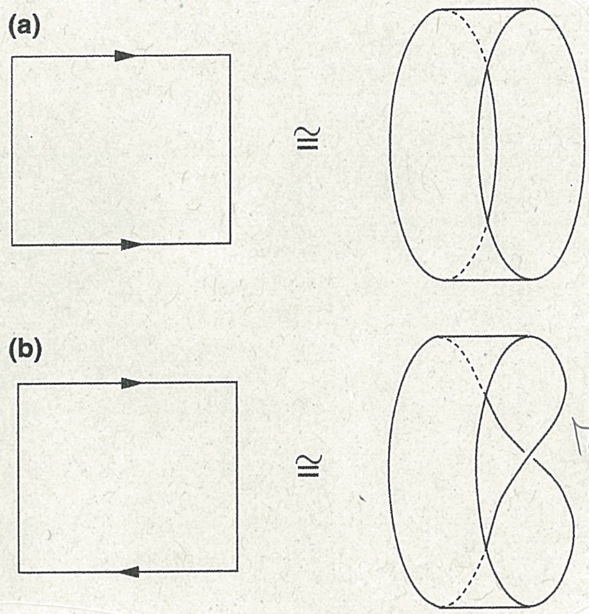
Cutting, pasting, quotienting



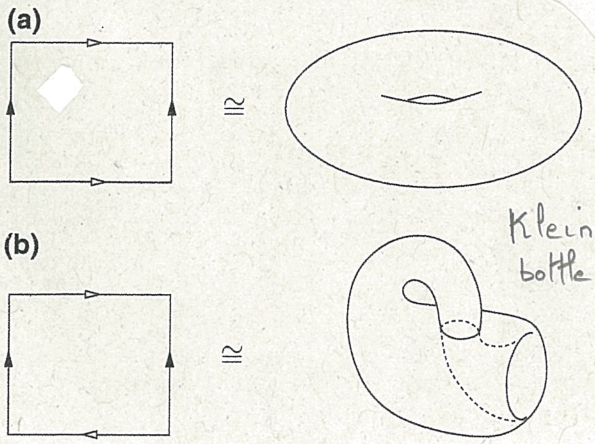
Cut and paste.



Quotient (= identification)



Möbius strip



Klein bottle

⚠ This operation (quotient) does not always produce a topological manifold.

Real projective space $P^n(\mathbb{R})$:

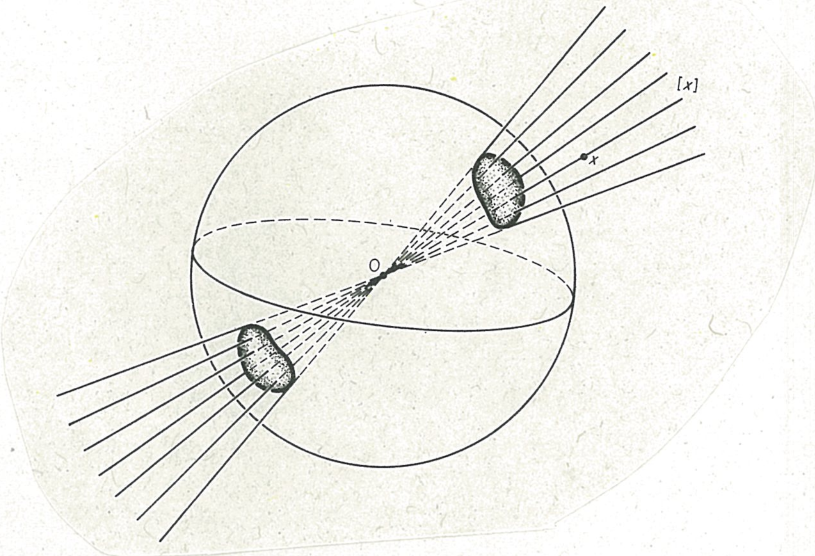
Consider $X := \mathbb{R}^{n+1} \setminus \{0\}$ and set

$(x_1, \dots, x_{n+1}) \sim (y_1, \dots, y_{n+1})$ if $\exists\, t \in \mathbb{R}^*$ s.t.

$(y_1, \dots, y_{n+1}) = (t x_1, \dots, t x_{n+1})$.

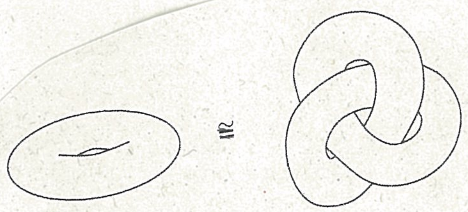
Then $P^n(\mathbb{R}) := X/\sim$

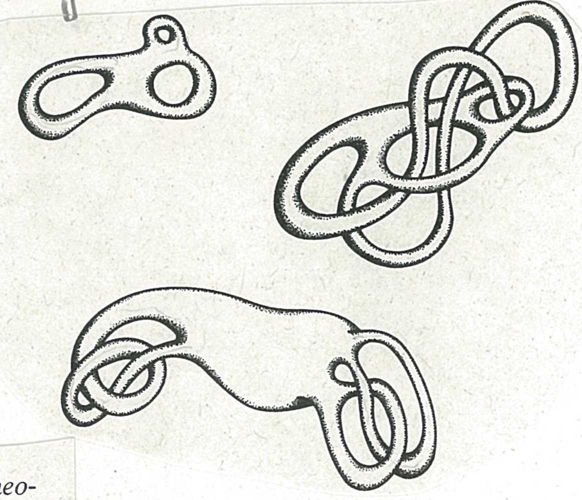
The equivalence class $[x]$ can be visualized as lines through the origin.



Homeomorphic topological manifolds

if there exists a homeomorphic function between them





Theorem *Every compact, connected, orientable 2-manifold is homeomorphic to a sphere with handles added. Two such manifolds with the same number of handles are homeomorphic and conversely, so that the number of handles (called the genus) is the only topological invariant.*

[Boothby, p 14]