Lattices and Poisson's Summation Formula (PSF).

Let $\{a_j\}_{j=1}^{d}$ be a basis of $\mathbb{R}^d$.

Def. A lattice $\Gamma \subset \mathbb{R}^d$ with a basis $\{a_j\}_{j=1}^{d} \subset \mathbb{R}^d$ is defined by

$$\Gamma := \{ n_1 a_1 + n_2 a_2 + \cdots + n_d a_d \mid n_1, \cdots, n_d \in \mathbb{Z} \}$$

$$= \mathbb{Z} a_1 \oplus \mathbb{Z} a_2 \oplus \cdots \oplus \mathbb{Z} a_d$$

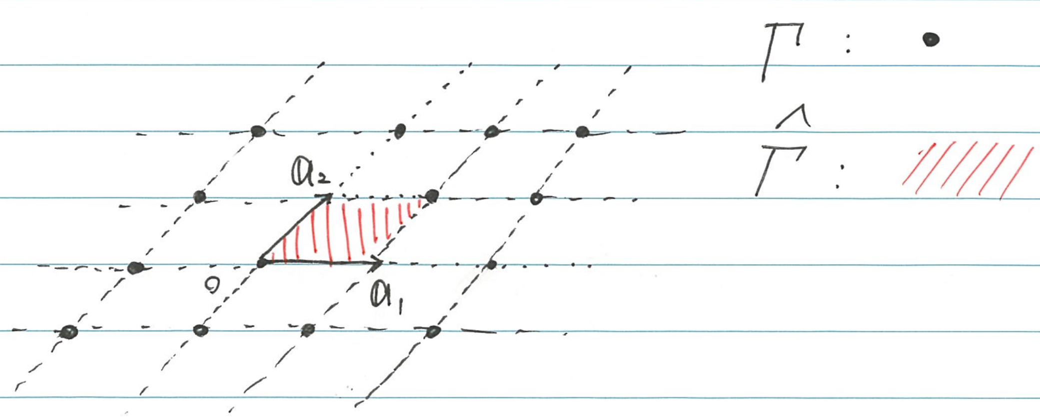
(free $\mathbb{Z}$-module with a basis $\{a_j\}$).

We choose a fundamental domain (unit cell)

$$\hat{\Gamma} := \{ t_1 a_1 + \cdots + t_d a_d \mid t_1, \cdots, t_d \in [0,1) \}.$$

↖ n-parallelepiped.

∘ In the case $d = 2$:



Properties.

(i) Γ is a subgroup of $\mathbb{R}^d$: $\forall a, a' \in \Gamma$

- $a + a' \in \Gamma$,
- $-a \in \Gamma$,
- $0 \in \Gamma$.

Moreover, $\Gamma + a = \Gamma$.

(ii) (For Tsuzu-san)

There are isoms of LCAGs :

$$\Gamma \cong \mathbb{Z}^d,$$

$$\mathbb{R}^d/\Gamma \cong \mathbb{T}^d \quad (\text{with } \mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\})$$

(iii) ~~Let~~ Set $A = (a_1 \cdots a_d) \in M_d(\mathbb{R})$.

Then, $|\hat{\Gamma}| = |\det(a_1 \cdots a_d)|$.

(iv) $$\mathbb{R}^d = \bigsqcup_{a \in \Gamma} (\hat{\Gamma} + a)$$

a_j^*

- **Dual lattice Γ^***

For a lattice $\Gamma = \mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_d$,

$$\Gamma^* := \{ \mathbb{k} \in \mathbb{R}^d \mid \forall a \in \Gamma,\ \mathbb{k} \cdot a \in \mathbb{Z} \}$$
$$= \{ \mathbb{k} \in \mathbb{R}^d \mid \forall a \in \Gamma,\ e^{2\pi i \mathbb{k} \cdot a} = 1 \}$$

($2\pi\Gamma^*$ is called reciprocal lattice).

Then, Γ^* is again a lattice in $\mathbb{R}^d$.

$\because$ Let $\{a_j^*\}_{j=1}^d \subset (\mathbb{R}^d)^*$ be the dual basis of $\{a_j\}_{j=1}^d$, i.e., $a_j^*(a_k) = \delta_{jk}$.

$\rightsquigarrow$ $\exists! \{\mathbb{k}_j\}_{j=1}^d \subset \mathbb{R}^d$ s.t. $a_j^* = \mathbb{k}_j \cdot$

$\rightsquigarrow$ $\Gamma^* = \mathbb{Z}\mathbb{k}_1 \oplus \cdots \oplus \mathbb{Z}\mathbb{k}_d$. $\square$

Exercise

(i) Show

(ii) $d = 3$. (iii) represent $\mathbb{k}_j$ in $d = 2$.

(iv) $K = (\mathbb{k}_1 \cdots \mathbb{k}_d) \in M_d(\mathbb{R}) \rightsquigarrow \det K = (\det A)^{-1}$

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Fourier analysis on Γ.

Def. A function $f : \mathbb{R}^d \to \mathbb{C}$ is Γ-periodic if $f(x + \mathbb{a}) = f(x)$ for $\forall x \in \mathbb{R}^d$ and $^\forall \mathbb{a} \in \Gamma$.

Example. For each $\mathbb{k} \in \Gamma^*$,

$$e_\Gamma^{(\mathbb{k})}(x) := \exp(2\pi i \mathbb{k} \cdot x) \quad \forall x \in \mathbb{R}^d$$

defines a Γ-periodic function $e_\Gamma^{(\mathbb{k})}$.

Moreover, for any smooth Γ-periodic function $f : \mathbb{R}^d \to \mathbb{C}$, the Fourier series

$$f(x) = \sum_{\mathbb{k} \in \Gamma^*} a_{\mathbb{k}} e_\Gamma^{(\mathbb{k})}(x)$$

converges uniformly, where

$$a_{\mathbb{k}} = \frac{1}{|\mathcal{X}_\Gamma|} \int_{\mathcal{X}_\Gamma} \overline{e_\Gamma^{(\mathbb{k})}(x)}\, f(x)\, dx.$$

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Def. and Prop.

We define the Γ-periodization $P_\Gamma f : \mathbb{R}^d \to \mathbb{C}$ of $f \in \mathcal{S}(\mathbb{R}^d)$ by

$$[P_\Gamma f](x) := \sum_{a \in \Gamma} f(x+a) \quad \forall x \in \mathbb{R}^d.$$

Then, $P_\Gamma f \in C^\infty(\mathbb{R}^d)$ and Γ-periodic.

Proof.

- Γ-periodicity : ok.
- smoothness : $\forall \alpha \in \mathbb{N}^d, \forall N \in \mathbb{N}\ \exists C_{\alpha,N} > 0$ s.t. $|\partial_x^\alpha f(x)| \leq C_{\alpha,N} (1+|x|)^{-N}$.

$$\sup_{x \in \mathbb{R}^d} \sum_{a \in \Gamma} |\partial_x^\alpha f(x+a)|$$

$$\leq C_{\alpha,N} \left(\underbrace{\sup_{x \in \mathbb{R}^d} \sum_{a \in \Gamma} (1+|x+a|)^{-N}}_{< \infty \text{ if } N > d+1.} \right)$$

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PSF. Let $f \in \mathcal{S}(\mathbb{R}^d)$ and $\mathbb{k} \in \Gamma^*$.

$$\hat{f}(\mathbb{k}) = \int_{\mathbb{R}^d} e^{-2\pi i \mathbb{k}\cdot x} f(x)\,dx$$

Fourier transform.

$$= \sum_{a \in \Gamma} \int_{\hat{\Gamma}+a} e^{-2\pi i \mathbb{k}\cdot x} f(x)\,dx$$

$$= \sum_{a \in \Gamma} \int_{\hat{\Gamma}} e^{-2\pi i \mathbb{k}\cdot (x+a)} f(x+a)\,dx$$

$$= \int_{\hat{\Gamma}} e^{-2\pi i \mathbb{k}\cdot x} \left(\sum_{a\in\Gamma} f(x+a) \right) dx$$

$$= \frac{1}{|\hat{\Gamma}|} \int_{\hat{\Gamma}} e^{-2\pi i \mathbb{k}\cdot x} \left(|\hat{\Gamma}| \sum_{a\in\Gamma} f(x+a) \right) dx$$

$\mathbb{k}$-th Fourier coeff. of $|\hat{\Gamma}| P_\Gamma f$

Theorem For any $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\sum_{\mathbb{k}\in\Gamma^*} \hat{f}(\mathbb{k})\, e_\Gamma^{(\mathbb{k})}(x) = |\hat{\Gamma}| \sum_{a\in\Gamma} f(x+a)$$

(uniformly)

In particular,

$$\sum_{\mathbb{k}\in\Gamma^*} \hat{f}(\mathbb{k}) = |\hat{\Gamma}| \sum_{a\in\Gamma} f(a).$$

A invertible $\rightsquigarrow \det(VAV^{-1}) = \det\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} = \underline{x_1 \cdots x_n}$

$[0, x_1] \times \cdots \times [0, x_n]$ の体積.

<u>Cor.</u> For any lattice $\Gamma \subset \mathbb{R}^d$,

$$\hat{\delta}_{\Gamma^*} = \frac{1}{|\hat{\Gamma}|}\delta_{\Gamma} \quad \text{in } \mathcal{S}'(\mathbb{R}^d).$$

In particular, $\Gamma = \mathbb{Z}^d$, then

$$\hat{\delta}_{\mathbb{Z}^d} = \delta_{\mathbb{Z}^d}.$$

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