《Fourier Analysis》多元数理 STE-16-7-41222 by E.M.Stein & R.Shakarchi

§5.1.6 Proposition 1.11 If $f, g \in S(\mathbb{R})$, then

(i) $f*g \in S(\mathbb{R})$; (ii) $f*g = g*f$; (iii) $\widehat{(f*g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

Proof. Recall: $f*g(x) := \int_{\mathbb{R}} f(y)g(x-y)\,dy$,

(#) $S(\mathbb{R}) := \{f \mid \sup_{x\in\mathbb{R}} |x|^k |f^{(l)}(x)| < \infty,\ \forall k, l \geq 0\}$,

$\hat{f}(\xi) := \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi}\,dx$.

(i) Firstly prove that $\forall l \geq 0$, $\sup_x |x|^l |g(x-y)| \leq A_l (1+|y|)^l$: (A_l is a const)

$|x|^l|g(x-y)| = |x-y+y|^l|g(x-y)| \leq (|x-y|+|y|)^l |g(x-y)|$ ①

If $|x-y| \leq |y|$, $(|x-y|+|y|)^l \leq (2|y|)^l = 2^l|y|^l$;

If $|x-y| \geq |y|$, $(|x-y|+|y|)^l \leq (2|x-y|)^l = 2^l|x-y|^l$.

$\therefore$ ① $\leq 2^l(|x-y|^l + |y|^l)|g(x-y)| = 2^l(|x-y|^l|g(x-y)| + |y|^l|g(x-y)|)$ ②

$\because g \in S(\mathbb{R})$ $\therefore \sup_{\mathbb{R}}(|x-y|^l|g(x-y)|) < +\infty$, $\sup_{\mathbb{R}}|g(x-y)| < +\infty$

$\therefore$ ② $\leq 2^l(C_1 + C_2|y|^l) \leq C_3(1+|y|^l)$, $C_3 := 2^l \max\{C_1, C_2\}$

$\leq C_3(1+|y|)^l + C_3 \leq 2C_3(1+|y|)^l$

$= A_l(1+|y|)^l$, $A_l := 2C_3 = 2^{l+1}\max\{C_1, C_2\}$

$\therefore \sup_x |x|^l|g(x-y)| \leq$ ① $\leq$ ② $\leq A_l(1+|y|)^l$.

Then, ~~integrate each side at $\mathbb{R}$ by y~~. $\forall l > 0$,

~~$\sup_x |x|^l \int_{\mathbb{R}} |g(x-y)|dy \leq A_l \int_{\mathbb{R}} (1+|y|)^l dy$~~

$\sup_x |x|^l \int_{\mathbb{R}} |f(y)g(x-y)|dy \leq A_l \int_{\mathbb{R}} |f(y)|(1+|y|)^l dy$

$\sup_x |x^l (f*g)(x)| \leq A_l\left[\int_{-1}^{1} |f(y)|\,2^l dy + \int_{|y|>1} |f(y)|(2|y|)^l dy\right]$

$= 2^l A_l\left[\int_{-1}^{1} |f(y)|dy + \int_{|y|>1} |f(y)||y|^l dy\right]$

$< +\infty$, because $f(y) \in S(\mathbb{R})$

Then prove $\sup_x |x^l (\frac{d}{dx})^k (f*g)(x)| < +\infty$, for any $l \geq 0$, $k \in \mathbb{N}$;

$\because g \in S(\mathbb{R})$ $\therefore \forall k \in \mathbb{N}$, $g^{(k)} \in S(\mathbb{R})$

$\therefore$ Using the same argument, $\sup_x |x^l (f*g^{(k)})(x)| < +\infty$

$\because (\frac{d}{dx})^k (f*g)(x) = (\frac{d}{dx})^k \int_{\mathbb{R}} f(y)g(x-y)dy = \int_{\mathbb{R}} f(y) g^{(k)}(x-y)dy = (f*g^{(k)})(x)$

$\therefore \sup_x |x^l (\frac{d}{dx})^k (f*g)(x)| < +\infty$ for any $l \geq 0$, $k \in \mathbb{N}$

$\therefore f*g \in S(\mathbb{R})$

(ii) $(f*g)(x) = \int_{\mathbb{R}} f(y)g(x-y)\,dy = \int_{\mathbb{R}} f(-y)g(x+y)\,dy$

(because if $\int_{\mathbb{R}} F(\underset{y}{\cancel{x}})\,d\underset{y}{\cancel{x}}$ exists, then $\int_{\mathbb{R}} F(\underset{y}{\cancel{x}})\,d\underset{y}{\cancel{x}} = \int_{\mathbb{R}} F(-\underset{y}{\cancel{x}})\,d\underset{y}{\cancel{x}}$)

Let $u := x+y$, then $du = dy$, $\int_{-\infty}^{\infty} G(y)\,dy = \int_{-\infty+x}^{\infty+x} G(u)\,du = \int_{-\infty\cancel{+x}}^{\infty\cancel{+x}} G(u)\,du$

$(f*g)(x) = \int_{\mathbb{R}} f(x-u)g(u)\,du = (g*f)(x)$.

(iii) Consider $F(x,y) := f(y)g(x-y)e^{-2\pi i x\xi}$.

$F_1(x) := \int_{\mathbb{R}} F(x,y)\,dy = [\int_{\mathbb{R}} f(y)g(x-y)\,dy]\,e^{-2\pi i x\xi} = (f*g)(x)e^{-2\pi i x\xi}$

$F_2(y) := \int_{\mathbb{R}} F(x,y)\,dx = f(y)\int_{\mathbb{R}} g(x-y)e^{-2\pi i x\xi}\,dx$ #

$= f(y)e^{-2\pi i y\xi}\int_{\mathbb{R}} g(x-y)e^{-2\pi i(x-y)\xi}\,d(x-y) = f(y)e^{-2\pi i y\xi}\hat{g}(\xi)$

$\because \widehat{(f*g)}(\cancel{x}\xi) = \int_{\mathbb{R}} (f*g)(x)e^{-2\pi i x\xi}\,dx = \int_{\mathbb{R}} F_1(x)\,dx = \iint_{\mathbb{R}^2} F(x,y)\,dx\,dy$,

$\hat{f}(\xi)\cdot\hat{g}(\xi) = [\int_{\mathbb{R}} f(y)e^{-2\pi i y\xi}\,dy]\,\hat{g}(\xi) = \int_{\mathbb{R}} F_2(y)\,dy = \iint_{\mathbb{R}^2} F(x,y)\,dx\,dy$

$\therefore \widehat{(f*g)}(\xi) = \hat{f}(\xi)\cdot\hat{g}(\xi)$.