Summary

$M(\mathbb{R}^d) :=$ set of all Radon measures on $\mathbb{R}^d$

$\mu \in M(\mathbb{R}^d) \Leftrightarrow$ 1) μ is a Boral measure $(\Omega, \mathcal{F}, \mathbb{P})$

2) μ is locally finite, i.e.

$\forall x \in \mathbb{R}^d\ \exists$ compact $K \ni x$, $\mu(K) < \infty$

3) μ is inner regular, i.e.

$\forall$Borel set A, $\mu(A) = \sup_{\substack{K \subset A \\ \text{compact}}} \mu(K)$

compact $\Leftrightarrow$ closed && bounded

$M(\mathbb{R}^d) =$ dual of $C_c(\mathbb{R}^d)$, i.e.

$\forall f \in C_c(\mathbb{R}^d)$, $\int f d\mu$ is meaningful

For $\mu \in M(\mathbb{R}^d)$, we set

$\gamma = \lim_{L \to \infty} \frac{1}{L^d} \mu_L * \widetilde{\mu_L}$ in the vague topology (γ is called autocorelation)

Remark: γ does not see the details of μ

Proposition: If μ is translation bounded then

$\exists$ at least one autocorelation which is

translation bounded and positive definite $\gamma(f * \tilde{f}) \geq 0$

Thm (Extension of Bochner)

A measure γ on $\mathbb{R}^d$ is positive definite iff

$\exists \hat{\gamma} \in \Pi(\mathbb{R}^d)$ which is positive, and satisfies

$\int f * \tilde{f}(x)\, \gamma(dy) = \int |\hat{f}(x)|^2\, \hat{\gamma}(dx)\quad \forall f \in C_c(\mathbb{R}^d)$

Remark

$\hat{\gamma} = \hat{\gamma}_{pp} + \hat{\gamma}_{ac} + \hat{\gamma}_{sc}$

Remark: when [BG] speaks about thin decomposition,

they assume $\hat{\gamma}$ is a regular measure (= Radon + outer regular)

Comparison with Hof

Consider $g \in L^1(\mathbb{R}^d)$ $\quad L^1(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{R} \mid \int_{\mathbb{R}^d} |f(x)|\,dx < \infty\}$

then

$$\begin{tikzcd} g \arrow[r, "*"] \arrow[d, "\mathcal{F}"'] & g * \tilde{g} \arrow[d, "\mathcal{F}"] \\ \hat{g} \arrow[r, "|\cdot|^2"'] & |\hat{g}|^2 \end{tikzcd} \qquad \begin{tikzcd} \mu \arrow[r, "*"] \arrow[dr, "\text{«Hof»}"'] & \gamma = \mu \circledast \tilde{\mu} \arrow[d, "\wedge", "\text{«BG»}"'] \\ & \hat{\gamma} \end{tikzcd}$$

Lemma

Let μ be a tempered measure s.t. $\hat{T}_\mu = T_{\hat{\mu}}$

Then $\hat{\mu}(\{\xi\}) = \lim_{n\to\infty} \mu(\phi_n)$ with $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R}^d)$ satisfying

1) $|\phi_n| \leq f \in L^1(\hat{\mu})\ \forall n \Rightarrow$ Lebesgue dominated convergence thm
2) $\phi_n(\xi) = 1$
3) $\lim_{n\to\infty} \phi_n(t) = 0 \quad \forall t \neq \xi$

Thm. Let μ be a translation bounded measure, and suppose $\hat{\mu}$ is also translation bounded, then

$$\hat{\mu}(\{\xi\}) = \lim_{n\to\infty} \frac{1}{n^d} \int_{C_n + a_n} e^{-2\pi i \xi\cdot x} \mu(dx) \quad \forall \xi \in \mathbb{R}^d \ \forall \text{sequence } \{a_n\} \subset \mathbb{R}^d$$

$\uparrow$ CUBE

Idea of the proof

Use $\hat{\phi}_n(x) = n^{-d} e^{-2\pi i \xi\cdot x} \chi_{C_n + a_n}(x)$

$\Leftrightarrow \phi_n(y) = n^{-d} e^{-2\pi i a_n \cdot y} \prod_{j=1}^{d} \frac{\sin(\pi n(\xi - y)_j)}{\pi(\xi - y)_j}$

Apparently condition 1 is not satisfied.

Since $\chi_{C_n + a_n}$ is not smooth, we'll make it more smooth.

Consider $\omega \in C_c^\infty(\mathbb{R}^d)$, supp $\omega \in B(0,1)$, $\omega > 0$, $\omega(-x) = \omega(x)$, $\int_{\mathbb{R}^d} \omega(x)\,dx = 1$

$\forall \varepsilon > 0$, we set $\omega_\varepsilon(x) = \varepsilon^{-d} \omega(x/\varepsilon)$

Then $\omega_\varepsilon \in C_c^\infty(\mathbb{R}^d)$, supp $\omega_\varepsilon \in B(0,\varepsilon)$, $\int_{\mathbb{R}^d} \omega_\varepsilon(x)\,dx = 1$

Then we set $\hat{\phi}_n^\varepsilon := \omega_\varepsilon * \hat{\phi}_n \in C_c^\infty(\mathbb{R}^d)$

(called mollification)

At the end, control the behavior as $\varepsilon \searrow 0$

Recall $\hat{\gamma} = \hat{\gamma}_{pp} + \hat{\gamma}_{ac} + \hat{\gamma}_{sc}$, but $\gamma_{ac}(\{\xi\}) = \gamma_{sc}(\{\xi\}) = 0$

$\therefore \hat{\gamma}(\{\xi\}) = \hat{\gamma}_{pp}(\{\xi\})$

Thm : Let μ be a translation bounded measure with unique autocorelation γ

Suppose that for any $\xi \in \mathbb{R}^d$

$$m_\xi := \lim_{L\to\infty} \frac{1}{L^d} \int_{C_L+a} e^{-2\pi i \xi\cdot x} \mu(dx)$$

exists, uniformly in $a \in \mathbb{R}^d$. (Not depending on a)

Then $\hat{\gamma}(\{\xi\}) = |m_\xi|^2$

Idea of proof

$$\hat{\gamma}(\{\xi\}) = \lim_{M\to\infty} \frac{1}{M^d} \int_{C_M} e^{-2\pi i \xi\cdot x} \gamma(dx)$$

$$\hookrightarrow = \lim_{\cancel{N}\to\infty\ L} \frac{1}{\cancel{N}^d\ L} \int \mu_L * \tilde{\mu}_L$$

$$= \lim_{\substack{M\to\infty \\ L\to\infty}} M^{-d} L^{-d} \iint \chi_{C_M}(x+y) e^{-2\pi i \xi (x+y)} \mu_L(dx)\, \tilde{\mu}_L(dy)$$

Example 9.1 on P336

9.2 P343

9.3 P344 FOR ME

9.5 P346

9.6 P351