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<u>Reminder</u>. Let $\mu : \mathcal{A}_B \to \mathbb{R}$ be a measure

1, μ is translation bounded if $\forall$ K compact of $\mathbb{R}^n$
$\sup_{x \in \mathbb{R}^n} |\mu|(K+x) = \alpha_K < \infty$

2, μ is slowly increasing if $\int_{\mathbb{R}^n} (1+|x|)^{-k} |\mu|(dx) < \infty$

3, $\tilde{\phi} = \overline{\phi(-\cdot)}$, $\tilde{\mu}(\phi) = \overline{\mu(\tilde{\phi})}$

<u>Remark</u>:

1, If μ is translation bounded, then
$|\mu|(A) \leq \alpha_1 \lambda(A + C_1)$ for any $A \subset \mathbb{R}^n$, measurable

2, If μ is translation bounded, then μ is slowly increasing (choose $k = d+2$)

<u>Recall</u>: μ has an autocorrelation if
$\lim_{L\to\infty} \frac{1}{L^d} \mu_L * \tilde{\mu}_L$ has a limit in the vague topology.
$\mu_L \leftarrow \mu \chi_{C_L}$
The limit is denoted by γ

<u>Proposition</u>: Let μ be translation bdd with a unique autocorrelation γ_μ. Let $D \subset \mathbb{R}^d$ s.t

$$\lim_{L\to\infty} \frac{1}{L^d} \lambda(D_r \cap C_L) = 0 \quad \forall r > 0.$$

Let ν be a translation bdd measure with support in D
Then $\gamma_{\mu+\nu} = \gamma_\mu$

<u>Proof</u>: $(\mu+\nu)_L * \widetilde{(\mu+\nu)_L}$

$$= (\mu_L + \nu_L) * (\tilde{\mu}_L + \tilde{\nu}_L) = \mu_L * \tilde{\mu}_L + \mu_L * \tilde{\nu}_L + \nu_L * \tilde{\mu}_L + \nu_L * \tilde{\nu}_L$$

Let us show that $\lim_{L\to\infty} \frac{1}{L^d} \mu_L * \tilde{\nu}_L = 0$

One has $|\mu_L * \tilde{\nu}_L(\phi)| \leq \int |\mu_L|(dx) \int |\tilde{\nu}_L|(dy) |\phi(x+y)|$

and: $\int_{\mathbb{R}^d} |\tilde{\nu}_L|(dy) |\phi(x+y)|$
$\leq \|\phi\|_\infty \int_{\mathbb{R}^d} |\tilde{\nu}_L|(dy) \chi_A(x+y)$ with $A = \operatorname{supp} \phi$

$$= \|\phi\|_\infty \int_{\mathbb{R}^d} |\tilde{\nu}_L|(dy)\ \chi_{A-x}(y)$$

$$= \|\phi\|_\infty \int_{\mathbb{R}^d} |\nu_L|(dy)\ \chi_{A-x}(-y)$$

$$= \|\phi\|_\infty \int_D |\nu_L|(dy)\ \chi_{x-A}(y) \quad (2)$$

$$\leq \underbrace{\alpha_1 C_A}_{\alpha} < \infty \text{ and independent of } x$$

but $(2) = 0$ if $x - A \cap D = \emptyset$

$\Leftrightarrow x \notin A + D$

It means that: $\int |\tilde{\nu}_L|(dy)\ |\phi(x+y)| \leq \alpha \chi_{A+D}$

Then $\frac{1}{L^d}\int \mu_L(dx) \int |\tilde{\nu}_L|(dy)\ |\phi(x+y)|$

$$\leq \frac{1}{L^d}\ \alpha \int |\mu_L(dx)|\ \chi_{A+D}(x)$$

$$= \frac{1}{L^d}\ \alpha \int |\mu|(dx)\ \chi_{(A+D)\cap C_L}(x)$$

$$\leq \frac{1}{L^d}\ \beta\ \lambda\big((D+A)\cap C_L + C_1\big)$$

$$\leq \beta_1 \frac{\lambda(D_r \cap C_{L+2})}{(L+2)^d} \frac{(L+2)^d}{L^d} \xrightarrow{L\to\infty} 0$$

Reminder:

1, A measure is tempered if $\int |\mu|\, dx\, f(x) < \infty\ \forall f \in S(\mathbb{R}^d)$

Then we write: $\mu(f) \equiv T_\mu(f) = \int_{\mathbb{R}^n} \mu(dx)\, f(x)$

Then $\mathcal{F}\, T_\mu(f) := T_\mu(\hat{f})$

2, If μ is slowly increasing then μ is a tempered measure

3, A tempered measure μ is positive definite if

$\mu(\phi * \tilde{\phi}) \geq 0 \quad \forall \phi \in S(\mathbb{R}^d)$

Lemma: Every autocorrelation which is tempered is positive definite.

Proof: It is sufficient to show that: $(\mu * \tilde{\mu})(\phi * \tilde{\phi}) \geq 0$

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One has:

$$(\mu * \tilde{\mu})(\phi * \tilde{\phi})$$

$$= \int \mu(dx) \int \tilde{\mu}(dy) \left(\phi * \tilde{\phi}\,(x+y)\right)$$

$$= \int \mu(dx) \int \tilde{\mu}(dy) \underbrace{\int \phi(z)\, \tilde{\phi}(x+y-z)\, dz}_{= \phi(z+x)\, \tilde{\phi}(y-z)\, dz}$$

$$= \int dz \left[\int \mu(dx)\, \phi(z+x) \right] \underbrace{\left[\int \tilde{\mu}(dy)\, \tilde{\phi}(y-z) \right]}_{= \overline{\int \mu(dy)\, \phi(z+y)}}$$

$$= \int dz \left[\int \mu(dx)\, \phi(x+z) \right] \overline{\left[\int \mu(dx)\, \phi(x+z) \right]}$$

$$= \int dz \left| \int \mu(dx)\, \phi(x+z) \right|^2 \geq 0$$

<u>Thm</u>: A measure μ is a positive definite tempered measure iff μ is the Fourier transform of a positive slowly increasing measure

<u>Lemma</u>: Let μ be a tempered measure s.t $\widehat{T_\mu} = T_{\hat{\mu}}$ with $\hat{\mu}$ a tempered measure.

Then $\hat{\mu}(\{\xi\}) = \lim_{n\to\infty} \mu(\phi_n)$ for any sequence

$\{\phi_n\} \subset \mathcal{S}(\mathbb{R}^n)$ s.t

1) $|\phi_n| \leq f \in L^1(\hat{\mu})\ \forall n$

2) $\varphi_n(\xi) = 1$

3) $\lim_{n\to\infty} \phi_n(t) = 0 \ \forall t \neq \xi$

<u>Proof</u>: Consider $\hat{\mu} - \hat{\mu}(\xi)\,\delta_\xi$ is a continuous measure at ξ

then $\lim_{n\to\infty} \left(\hat{\mu} - \hat{\mu}(\{\xi\})\,\delta_\xi\right)(\phi_n)$ $\quad (f \in L^1(\hat{\mu} - \hat{\mu}(\xi)\delta_\xi))$

$$= \lim_{n\to\infty} \left(\hat{\mu}(\phi_n) - \hat{\mu}(\{\xi\}) \underbrace{\phi_n(\xi)}_{\downarrow_{n\to\infty}\, 1} \right)$$

$$= 0$$

Lebesgue dominated convergence thm.