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Measure + Integration

I/ Positive valued measures

Let X be a space (set), and let $\mathcal{P}(X)$ be the set of all subsets of X

Roughly: a measure μ on X is a map $\mathcal{P}(X) \to [0, \infty]$ s.t.

$$\mu(\bigcup_j E_j) = \sum_{j \in \mathbb{N}} \mu(E_j).$$

whenever $E_j \cap E_k = \emptyset$ and $E_j \in \mathcal{P}(X)$

Def: Let $\mathcal{A} \subset \mathcal{P}(X)$, $\mathcal{A} \neq \emptyset$

1, $\mathcal{A}$ is an algebra if closed for finite union and complement:

$E_j \cup E_k \in \mathcal{A}$ whenever $E_j, E_k \in \mathcal{A}$.
$E_j^c := X \setminus E_j \in \mathcal{A}$.

2, $\mathcal{A}$ is a σ-algebra if closed for countable union and compliment.

3, $(X, \mathcal{A})$ with $\mathcal{A}$ a σ-algebra is called a measurable space

Remark:

1) $\Rightarrow$ closed for finite intersection

2, $\Rightarrow$ closed for countable intersection.

1 or 2) $\Rightarrow$ closed for relative complement. $E \setminus F \in \mathcal{A}$ when $E, F \in \mathcal{A}$

Remark:

If $\mathcal{E} \subset \mathcal{P}(X)$, there always exists a smallest σ-algebra $\mathcal{A}$ with $\mathcal{E} \subset \mathcal{A}$

Def: If X is a topological space and let $\mathcal{E} = \{\text{open sets}\}$ then smallest σ-algebra containing $\mathcal{E}$ is called the Borel σ-algebra, and is denoted by $\mathcal{A}_B$. Any element of $\mathcal{A}_B$ is called a Borel set.

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Example: $X = \mathbb{R}$, with $\mathcal{E} = \{(a,b) \mid a<b\}$

$[a,b]$, $(a,b]$, $(-\infty,a)$ or $(a,\infty) \in \mathcal{A}_B$.

Def A measure on a measurable space $(X, \mathcal{A})$ is a map $\mu : \mathcal{A} \to [0,\infty]$ s.t.

$$\mu(\emptyset) = 0$$

$$\mu(\bigcup E_j) = \sum_j \mu(E_j), \quad E_j \in \mathcal{A}, \ E_j \cap E_k = \emptyset$$

$(X, \mathcal{A}, \mu)$ is called a measure space

Properties:

1, If $E, F \in \mathcal{A}$, $E \subset F$, then $\mu(E) \leq \mu(F)$.

2, If $\{E_j\} \subset \mathcal{A}$, then $\mu(\bigcup_j E_j) \leq \sum_j \mu(E_j)$

3, If $\{E_j\} \subset \mathcal{A}$ with $E_j \subset E_{j+1}$ then $\mu(\bigcup_j E_j) = \lim_{j\to\infty} \mu(E_j)$

4, If $\{E_j\} \subset \mathcal{A}$ with $E_j \supset E_{j+1}$, then $\mu(\bigcap_j E_j) = \lim_{j\to\infty} \mu(E_j)$ and $\mu(E_k) < \infty$ for some k.

Def: 1, If $\mu(X) < \infty$, the measure is finite

2, If $X = \bigcup_j E_j$ with $\mu(E_j) < \infty$ the measure μ is σ-finite

3, If $\mathcal{A} = \mathcal{A}_B \equiv \mathcal{A}_B(X)$, then μ is called a Borel measure

Example:

1, L.m on $\mathbb{R}$ (or $\mathbb{R}^d$) is a Borel measure

2, δ_x on $\mathbb{R}$ defined by $\delta_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$ is a Borel measure.

3, Let $\omega \in L^1_{loc}(\mathbb{R})$ and $\omega \geq 0$, then

$\mu(E) := \int_E \omega(x)\,dx$ defines a Borel measure.

ω is called density.

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II / Integration

let $(X, \mathcal{A}, \mu)$ a measure space

let $\phi = \sum_{j=1}^{N} c_j \chi_{E_j}$ with $c_j \in \mathbb{C}$, $E_j \in \mathcal{A}$.

ϕ is called a simple function.

<u>Def</u>: If $c_j \geq 0$ then:

$$\mu(\phi) = \int \phi \, d\mu := \sum_{j=1}^{N} c_j \mu(E_j)$$

<u>Def</u>: let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces and let: $f : X \longrightarrow Y$. f is measurable if $f^{-1}(E) \in \mathcal{A} \quad \forall E \in \mathcal{B}$

In particular: 1, $f : (X, \mathcal{A}) \to (\mathbb{R}, \mathcal{A}_B)$ is measurable $= f^{-1}((a, b)) \in \mathcal{A}$.

2, $f : X \to \mathbb{C}$, f is measurable if its real part and imaginary part are measurable.

<u>Proposition</u>:

If $f : X \to \mathbb{C}$ is measurable, then $\exists \{\phi_n\}_{n \in \mathbb{N}}$ of simple function s.t $\phi_n \xrightarrow{n \to \infty} f$ point wise, $|\phi_n| \xrightarrow[n\to\infty]{} |f|$ and $\phi_n \xrightarrow{n\to\infty} f$ uniformly if f is bounded.

<u>Def</u>: for $f : X \to [0, \infty]$ measurable, we set:

$$\int_X f \, d\mu := \sup \left\{ \int \phi \, d\mu \mid \phi \text{ simple}, \ 0 \leq \phi \leq f \right\}$$

<u>Def</u>: let $f : X \to \mathbb{C}$ be measurable then f is integrable if $\int |f| \, d\mu < \infty$

We write $f \in \mathcal{L}^1(X, \mu)$

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If $f \in \mathcal{L}^1(X, \mu)$ we set

$$\int f \, d\mu = \int (\mathrm{Re} f)_+ \, d\mu - \int (\mathrm{Re} f)_- \, d\mu + i \left(\int (\mathrm{Im} f)_+ \, d\mu - \int (\mathrm{Im} f)_- \, d\mu \right)$$