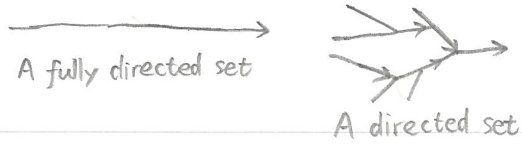

Def. A directed set is a set A with
a reflexive $(\alpha \leq \alpha)$ and transitive $\left(\begin{smallmatrix}\alpha\leq\beta\\ \beta\leq\gamma\end{smallmatrix} \Rightarrow \alpha\leq\gamma\right)$ binary relation $\leq$ such that
$\forall \alpha, \beta \in A, \exists \gamma \in A : \alpha \leq \gamma, \beta \leq \gamma$ (α, β can be uncomparable)

Ex. • $\mathbb{N}, \mathbb{R}$
• neighborhoods of $x \in X$, with $U \leq V \Leftrightarrow U \supset V$
• $\{P \mid$ partation of $[a,b]\} \ni P_1, P_2$, with $P_1 \leq P_2 \Leftrightarrow P_2$ is finer than P_2

Def. A net in X is a map $A \ni \alpha \mapsto X_\alpha \in X$ with A a directed set
(generalization of sequence)

Def: A net $\{X_\alpha\}_{\alpha\in A}$ in X converges to $x \in X$ ~~is~~ if
$\forall$ ngd. (neighborhood) U of x $\exists \alpha_0 \in A : \forall \alpha_0 \leq \alpha, x_\alpha \in U$
• A point $x \in \cancel{A}X$ is a cluster point for a net $\{x_\alpha\}_{\alpha\in A}$ if
$\forall U$ ngd. of x and for any $\alpha \in A$ $\exists \beta \in A$ with $\alpha \leq \beta$ and $x_\beta \in U$
Ex. $0, 1, 0, 1, 0, 1, \cdots$ (0, 1 are cluster points)

Prop.
Let $E \subset X$, then $x \in \overline{E}$ iff $\exists$ a net $\{x_\alpha\}_{\alpha\in A} \subset E$ converging to x,
and then $x \in E$ is an accumulation point of E is
$\exists$ a net $\{x_\alpha\}_{\alpha\in A}$ converging ~~to~~ in $E \setminus \{x\}$ to x

Remark: In metric spaces, a cluster point of a sequence is the limit of a convergent subsequence. Similarly, a cluster point of a net is the limit of a convergent subnet.

Def. A subnet is a composition of $A \ni \alpha \mapsto x_\alpha \in X$ with a new map
$B \ni \beta \mapsto \alpha_\beta \in A$ with B another directed set such
$\forall \alpha_0 \in A, \exists \beta_0 \in B$ with $\alpha_0 \leq \alpha_\beta$ for any $\beta_0 \leq \beta$

Recall that in $\mathbb{R}^n$, a set E is compact iff bounded & closed

Thm. If (X, ρ) is a metric space, and $E \subset X$, these are equivalent:
• E is complete and totally bounded $\Leftrightarrow$ ($\forall \varepsilon > 0$, E can be covered by a finite number of balls of radius ε.)
• Every sequence in E has a subsequence converging to a point in E
• Every open cover of E has a finite subcover (*
(For $X = \mathbb{R}^n$, they are equivalant to that E is bounded and closed)

In such a case we say that E is compact.

Def. In $(X, \mathcal{T})$, $E \subset X$ is compact if $(*)$:

every open cover of E admits a finite subcover.

Remark: If $f: X \to Y$ with $(X, \mathcal{T})(Y, \sigma)$ 2 top. space, is continuous and if $E \subset X$ is compact, then $f(E)$ is compact in Y.

Thm. Let $(X, \mathcal{T})$ be a top. space. A set $E \subset X$ is compact iff every net in E has a convergent subnet.