No.

Date

Topology

Reminder: $f: \mathbb{R}^n \to \mathbb{R}$ is continuous at $x \in \mathbb{R}^n$ if

$\forall \varepsilon > 0, \exists \delta > 0 : |f(x) - f(y)| < \varepsilon \ \forall \|x-y\| < \delta$

Def. A metric space (X, ρ) is a non-empty set X and map $\rho : X \times X \to [0, +\infty)$ such that:

1) $\rho(x,y) = \rho(y,x)$
2) $\rho(x,z) \le \rho(x,y) + \rho(y,z)$ (triangle inequality)
3) $\rho(x,y) = 0 \Leftrightarrow x = y$

ρ is called a metric.

Examples

1) $X = \mathbb{R}^n$, $\rho(x,y) = \|x-y\|_2 = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$

(the most normal one)

2) $X = C([a,b])$, $\rho(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)| \equiv \|f-g\|_\infty$

3) $X = C([a,b])$, $\rho(f,g) = \int_a^b |f(x) - g(x)|\,dx \equiv \|f-g\|_1$

~~4~~

4) X a curve in $\mathbb{R}^n$ parameterized by a diffeomorphism φ, ~~with~~ $\forall \varphi(t), \varphi(s) \in X$ with $t, s \in \mathbb{R}$, $\rho(\varphi(t), \varphi(s)) = \int_s^t \|\varphi'(z)\|\,dz$

5) X any non-empty set, $\rho(x,y) = \begin{cases} 0, & x = y \\ 1, & x \ne y \end{cases}$

For any metric space (X, ρ)

- $B(x, \delta) = \{y \in X \mid \rho(x,y) < \delta\}$
- $E \subset X$ is open ~~is~~ if $\forall x \in E, \exists \delta > 0 : B(x, \delta) \subset E$
- $E \subset X$ and $x \in X$ is an accumulation point of E if $B(x,\delta) \setminus \{x\} \cap E \ne \emptyset, \forall \delta > 0$.
- $E \subset X$ is closed if E contains all of its accumulation points.

Lemma (for any (X, ρ)) and series O_α and C_α

- If ~~a series~~ O_α is open for any α, then $\bigcup_\alpha O_\alpha$ is open, $\bigcap_{\alpha=1}^{N} O_\alpha$ is ope
- If C_α is closed for any α, then $\bigcap_\alpha C_\alpha$ is closed, $\bigcup_{\alpha=1}^{N} C_\alpha$ is closed

⚠ $\bigcap_{\alpha=1}^{\infty} O_\alpha$ may be closed

$\bigcup_{\alpha=1}^{\infty} C_\alpha$ may be open

More definitions for (X,ρ). Let $E \subset X$

- $Cl(E) \equiv \overline{E} :=$ the smallest closed set containing E

  $= E \cup \{$its accumulation points$\}$ (closure of E)

- E is dense in X $:\Leftrightarrow \overline{E} = X$

  Ex. $\overline{\mathbb{Q}} = \mathbb{R}$

- X is separable $:\Leftrightarrow$ it admits a countable dense set ($\supset$) ($\exists \overset{\text{bijective}}{f} : \mathbb{N} \to \bullet$)

- A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ converges to $x_\infty \subset X$

  if $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \rho(x_\infty, x_{\cancel{N} n}) < \varepsilon \ \forall n > N$

- A sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ is Cauchy

  if $\forall \varepsilon > 0, \exists N \in \mathbb{N} : \rho(x_m, x_n) < \varepsilon \ \forall m, n > N$

- X is complete if every Cauchy sequence is convergent in X

  Ex.

  - $\mathbb{R}^n$, $\rho(x,y) = \|x-y\|$ is complete
  - $C_c(\mathbb{R})$ with $\|\cdot\|_\infty$ is not complete; its completion is $C_0(\mathbb{R})$

    with bounded support $\qquad \{f \in C(\mathbb{R}) \mid \lim_{x\to\infty} f(x) = 0\}$

- $f : (X,\rho) \to (Y,\sigma)$ is continuous at $x \in X$

  if $\forall \varepsilon > 0 \ \exists \delta > 0 : \sigma(f(x), f(z)) < \varepsilon \ \forall z \in B(x,\delta)$

  f is continuous from X to Y if x is arbitrary.

Prop. 1 $(X,\rho), (Y,\sigma)$ 2 metric spaces, and $f : X \to Y$.

Then f is continuous from X to Y

iff $f^{-1}(U)$ is open in X whenever U is open in Y.

preimage of $U := \{x \in X \mid f(x) \in U\}$

Prop. 2 $E \subset X$, then $x \in \overline{E}$ iff $\exists \{x_n\} \subset E$ converging to x,

and x is an accumulating point of E iff $\exists \{x_n\} \subset E \setminus \{x\}$ converging to x

⚠ A metric isn't always ~~aw~~ available.

Sometimes $\triangle$ inequality is not satisfied, or $\rho(x,y) = 0$ when $x \neq y$.

\Tau

Def. A topological space $(X, \mathcal{T})$ is a non-empty set X and a collection $\mathcal{T} = \{T_\alpha\}_\alpha$ for $T_\alpha \subset X$ satisfying:

1) $\emptyset, X \in \mathcal{T}$
2) If $T_\alpha \in \mathcal{T}$, then $\bigcup_\alpha T_\alpha \in \mathcal{T}$
3) If $T_\alpha \in \mathcal{T}$, then $\bigcap_{\alpha=1}^{N} T_\alpha \in \mathcal{T}$

The elements of $\mathcal{T}$ are called the open sets

Example

1) If (X, ρ) a metric space, then $(X, \{\text{open sets}\})$ is a topological space.

Remarks: If X is a set and ε is an arbitrary collection of subsets of X, then we can always define a family $\mathcal{T} \supset \varepsilon$ such that $\{X, \mathcal{T}\}$ is a topological space.

- If $(X, \mathcal{T})$ is a topological space, and $Y \subset X$ we consider $\{T_\alpha \cap Y \mid T_\alpha \in \mathcal{T}\} =: \mathcal{T}_Y$ then $(Y, \mathcal{T}_Y)$ is a topological space (induced by $(X, \mathcal{T})$)

Some def. Let $(X, \mathcal{T})$ be a topological space

- If $x \in X$ and $E \subset X$, we say E is a neighborhood of x in X if $\exists T_\alpha \in \mathcal{T} : x \in T_\alpha \subset E$
- $E \subset X$ and $x \in X$ is an accumulation point of E if $\forall T_\alpha \in \mathcal{T}$ with $x \in T_\alpha$, one has $T_\alpha \cap E \neq \emptyset$
- E is closed if $E^c (:= X \setminus E)$ is open
- $f : (X, \mathcal{T}) \to (Y, S)$ is continuous if $\forall S_\alpha \in S : f^{-1}(S_\alpha) \in \mathcal{T}$

⚠ $(X, \mathcal{T})$ is Haustoff if $\forall x, y \in X$, $x \neq y$, $\exists U, V$ open $\subset X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$

$(X, \mathcal{T})$ is normal if $\forall E, F \subset X$, E, F closed, $E \cap F = \emptyset$, $\exists U, V$ open $\subset X$ such that $E \subset U$, $F \subset V$ and $U \cap V = \emptyset$

These 2 prop. are not automatric!

Lemma. If (X, ρ) is a metric space, $(X, \{\text{open sets}\})$ is normal & Haustoff

Thm. Let (X, J) be a topology space. They are equivallent:

- (X, J) is normal
- $\forall E, F \subset X$ closed and disjoint, $\exists f: X \to [0,1]$ continuous, with
  $\forall x \in E: f(x) = 0$; $\forall x \in F: f(x) = 1$ 

- $\forall E \subset X$ closed $\forall f: E \to \mathbb{R}$ continuous (with the topology inherited from X to E)
  $\exists \tilde{f}: X \to R$ continuous with $f = \tilde{f}$ on E (Tietze extention thm)

Remark

When (X, ρ) a metric space, we can consider $\{B(x, \frac{1}{n})\}$ convenient

A few more def. for (X, J)

- A neighborhood base for J at $x \in X$ is
  a family $\{U_\alpha\} \subset J$ with $x \in U_\alpha$, and $\forall E$ neighborhood of x $\exists \overset{\alpha}{\cancel{U_\alpha}} : x \in U_\alpha \subset$
- A base for J is a family $\{U_\alpha\} \subset J$ which
  contains a neighborhood base for any $x \in X$
- (X, J) is first countable if $\exists$ countable neighborhood base for any $x \in$
- second countable if $\exists$ countable base for J

Examples

- Metric spaces are first countable, $\{U_n\} = \{B(x, \frac{1}{n})\}$ for $n \in \mathbb{N}_+$
- If the metric space is separable, then it is second countable.
  Ex. $\mathbb{R}^n$ is second countable. Proof:
  $\mathbb{Q}^n$ is countable and $\overline{\mathbb{Q}^n} = \mathbb{R}^n \Rightarrow \mathbb{R}^n$ is separable $\Rightarrow 2^{nd}$ countable