Ex. 9.3 on P344 : ($\mathbb{Z}^2$-periodic point measures)

Let $\Gamma = \mathbb{Z}^2$ be the square lattice. Place a unit Dirac measure at each lattice point and a Dirac measure of complex weight κ at (a,b) and $(a,b)+\Gamma$.

Without loss of generality, assume $0 \leq a, b < 1$

This constitutes a $\mathbb{Z}^2$-periodic weighted Dirac comb ω: (see next page)

Recall: a measure ω is called Γ-periodic if $\forall x \in \Gamma : \delta_x * \omega = \omega$, we write

$左 = \int \delta_x(t) g(y-t)\,dt = \delta_x(g(y-\cdot)) = 右$

$左 = \iint f(s+t)\,d\delta_x(s)\,d\mu(t) = \int \delta_x(f(\cdot+t))\,d\mu(t) = \int f(x+t)\,d\mu(t) = 右$

$(\delta_x * g)(y) = g(y-x)$ for function g

$(\delta_x * \mu)(f) = \mu(f(x+\cdot))$ for measure μ

$\uparrow$

$\omega = \rho * \delta_{\mathbb{Z}^2}$, $\rho = \delta_{(0,0)} + \kappa\,\delta_{(a,b)}$ $\therefore \tilde{\rho}(f) = \overline{\rho(\tilde{f})} = \overline{\delta_{(0,0)}(\tilde{f}) + \kappa\,\delta_{(a,b)}(\tilde{f})} = \delta_{(0,0)}(f) + \bar{\kappa}\,\delta_{-(a,b)}(f)$

$\downarrow$ next page ①

Autocorrelation $\gamma_\omega = (\rho * \tilde{\rho}) * \delta_{\mathbb{Z}^2}$

$\Big[\rho * \tilde{\rho} = (\delta_{(0,0)} + \kappa\,\delta_{(a,b)}) * (\delta_{(0,0)} + \bar{\kappa}\,\delta_{-(a,b)})$

$\quad = (1+|\kappa|^2)\,\delta_{(0,0)} + \kappa\,\delta_{(a,b)} + \bar{\kappa}\,\delta_{-(a,b)}\Big]$

$\tilde{f} = \overline{f(-\cdot)}$, $\tilde{\rho} = \overline{\rho(\tilde{\cdot})}$

$$\begin{aligned}
\therefore \tilde{\rho}(f) = \overline{\rho(\tilde{f})} &= \overline{\delta_{(0,0)}(\tilde{f}) + \kappa\,\delta_{(a,b)}(\tilde{f})} \quad \text{Since } \overline{(a+bi)(c+di)} = (ac-bd)-(ad+bc)i = (a-bi)(c-di) \\
&= \overline{\delta_{(0,0)}(\tilde{f})} + \bar{\kappa}\,\overline{\delta_{(a,b)}(\tilde{f})} \\
&= \overline{\tilde{f}(0,0)} + \bar{\kappa}\,\overline{\tilde{f}(a,b)} = \overline{\overline{f(-0,-0)}} + \bar{\kappa}\,\overline{\overline{f(-a,-b)}} = f(0,0) + \bar{\kappa} f(-a,-b) \\
&= \delta_{(0,0)}(f) + \bar{\kappa}\,\delta_{-(a,b)}(f)
\end{aligned}$$

$f*g(x) = \int_{\mathbb{R}} f(y)\,g(x-y)\,dy$, $\mu * \nu(f) = \iint f(x+y)\,d\mu(x)\,d\nu(y)$, $\int f\,d\mu = \mu(f)$

$$\begin{aligned}
\therefore \rho * \tilde{\rho}(f) &= \iint_{\mathbb{R}^2} f(x+y)\,d\rho(x)\,d\tilde{\rho}(y) \\
&= \tilde{\rho}\left(\int_{\mathbb{R}} f(x+y)\,d\rho(x)\right) = \delta_{(0,0)}\left(\int_{\mathbb{R}} f(x+y)\,d\rho(x)\right) + \bar{\kappa}\,\delta_{-(a,b)}\left(\int_{\mathbb{R}} f(x+y)\,d\rho(x)\right) \\
&\overset{r=(a,b)}{=\!=} \int_{\mathbb{R}} f(x)\,d\rho(x) + \bar{\kappa}\int_{\mathbb{R}} f(x-r)\,d\rho(x) = \rho(f) + \bar{\kappa}\,\rho(f(\cdot-r)) \\
&= \delta_{(0,0)}(f) + \kappa\,\delta_{(a,b)}(f) + \bar{\kappa}\,\delta_{(0,0)}(f(\cdot-r)) + \kappa\bar{\kappa}\,\delta_{(a,b)}(f(\cdot-r)) \\
&= f(0) + \kappa f(r) + \bar{\kappa} f(-r) + |\kappa|^2 f(r-r) = (1+|\kappa|^2) f(0) + \kappa f(r) + \bar{\kappa} f(-r)
\end{aligned}$$

$\therefore \rho * \tilde{\rho} = (1+|\kappa|^2)\,\delta_{(0,0)} + \kappa\,\delta_{(a,b)} + \bar{\kappa}\,\delta_{-(a,b)}$

$\delta_{\mathbb{Z}^2} = \sum_{(x,y)\in\mathbb{Z}^2} \delta_{(x,y)}$

Autocorrelation: $\gamma_\omega = \lim_{L\to\infty} \frac{1}{L^2}\,\omega_L * \widetilde{\omega_L}$

$\widetilde{\omega_L} = \overline{\omega(\widetilde{x_L})} = \overline{\omega(\widetilde{x_L})} = \overline{\omega(x_L)} = \overline{\omega_L} = \overline{\rho * \delta_{\mathbb{Z}^2}}(x_L)$

$\omega_L * \widetilde{\omega_L} = \omega_L * \overline{\omega_L} = \big((\rho * \delta_{\mathbb{Z}^2}) * (\bar{\rho} * \delta_{\mathbb{Z}^2})\big)(x_L)$

$(\cdots)$

Shortcut:

$$\begin{aligned}
\rho * \tilde{\rho} &= (\delta_0 + \kappa\,\delta_r) * (\delta_0 + \bar{\kappa}\,\delta_{-r}) \\
&= \delta_{0+0} + \kappa\,\delta_{0+r} + \bar{\kappa}\,\delta_{0-r} + \kappa\bar{\kappa}\,\delta_{r-r} \\
&= (1+|\kappa|^2)\,\delta_0 + \kappa\,\delta_r + \bar{\kappa}\,\delta_{-r}
\end{aligned}$$

(since $\delta_a * \delta_b = \delta_{a+b}$)

$\gamma_\omega = (\rho * \tilde{\rho}) * \delta_{\mathbb{Z}^2}$, with

$\rho = \delta_0 + \kappa \delta_r$ ($r := (a,b) \in [0,1)^2$); $\rho(f) = f(0) + \kappa f(r)$;

$\tilde{\rho}(f) = \overline{\rho(\tilde{f})} = \overline{\tilde{f}(0) + \kappa \tilde{f}(r)} = \overline{\overline{f(0)}} + \overline{\kappa \overline{f(-r)}} = f(0) + \bar{\kappa} f(-r)$, so $\tilde{\rho} = \delta_0 + \bar{\kappa}\delta_{-r}$;

$(\rho * \tilde{\rho})(f) = \iint_{\mathbb{R}^2} f(x+y) d\rho(x) d\tilde{\rho}(y) = \int_{\mathbb{R}} \rho(f(\cdot + y))\, d\tilde{\rho}(y) = \int_{\mathbb{R}} (f(y) + \kappa f(y+r))\, d\tilde{\rho}(y)$

$= \tilde{\rho}(f) + \kappa \tilde{\rho}(f(\cdot + r)) = f(0) + \bar{\kappa} f(-r) + \kappa(f(r) + \bar{\kappa} f(r-r))$

$= f(0) + \bar{\kappa} f(-r) + \kappa f(r) + \kappa\bar{\kappa} f(0) = (1 + |\kappa|^2) f(0) + \bar{\kappa} f(-r) + \kappa f(r)$, so

$\rho * \tilde{\rho} = (1 + |\kappa|^2)\delta_0 + \bar{\kappa}\delta_{-r} + \kappa\delta_r$

Thm. Let Γ be a lattice in $\mathbb{R}^d$, and ω a Γ-invariant measure, represented as $\omega = \rho * \delta_\Gamma$ with ρ a finite measure.

Then the autocorrelation γ_ω of ω is given by

$$\gamma_\omega = (\rho * \tilde{\rho}) * \gamma_\Gamma = \mathrm{dens}(\Gamma)\,(\rho * \tilde{\rho}) * \delta_\Gamma$$

with the diffraction measure

☆ $$\widehat{\gamma_\omega} = \mathrm{dens}^2(\Gamma)\,|\hat{\rho}|^2 \delta_{\Gamma^*}$$

In particular, $\widehat{\gamma_\omega}$ is a positive pure point measure, with $\mathrm{supp}(\hat{\gamma}) \subset \Gamma^*$ and $\widehat{\gamma_\omega}(\{0\}) = \mathrm{dens}^2(\Gamma)\,|\hat{\rho}(0)|^2$

Continuing on Ex. 9.3

The corresponding diffraction measure is

$\widehat{\gamma_\omega} = |\hat{\rho}|^2 \delta_{\mathbb{Z}^2}$, with (for $t := (k,l) \in \mathbb{Z}^2$)

$\hat{\delta}_r(t) = \int_{\mathbb{R}^2} \delta_r(x) e^{-2\pi i \langle x,t \rangle} dx = \delta_r(e^{-2\pi i \langle \cdot, t \rangle}) = e^{-2\pi i \langle r,t \rangle}$

$\hat{\rho}(t) = \hat{\delta}_0(t) + \kappa \hat{\delta}_r(t) = e^0 + \kappa e^{-2\pi i \langle r,t \rangle} = 1 + \kappa e^{-2\pi i \langle r,t \rangle} = 1 + \kappa e^{-2\pi i (ak+bl)}$

The diffraction measure is supported on $\mathbb{Z}^2$ (which is self-dual), but need not have any non-trivial period. In fact, when $\kappa \neq 0$, $\widehat{\gamma_\omega}$ is crystallographic if both coordinates $a, b \in \mathbb{Q}$, and periodic (of rank 1) if precisely one of them $\in \mathbb{Q}$.

$$\widehat{\gamma_\omega} = \sum_{x \in \mathbb{Z}^2} |\hat{\rho}(x)|^2 \delta_x$$