Def. (FLC)

A point set (countable) $\Lambda \subset \mathbb{R}^d$ has finite local complexity (FLC) with respect to translations when

$\forall$ compact $K \subset \mathbb{R}^d : |\{(t+K) \cap \Lambda \mid t \in \mathbb{R}^d\}| < \infty$

Prop. A point set $\Lambda \subset \mathbb{R}^d$ has FLC $\Leftrightarrow$ $\Lambda - \Lambda$ is locally finite.

$\Lambda - \Lambda := \{x - y \mid x \in \Lambda, y \in \Lambda\}$

Def. (Dirac Comb) (P336)

Let Λ be a point set with FLC, then we Dirac Comb ω_Λ ~~$\omega : \Lambda \to \mathbb{C}$~~ is

$\omega_\Lambda := \sum_{x \in \Lambda} \omega(x) \delta_x$ where δ_x is Dirac measure, $\omega : \Lambda \to \mathbb{C}$ bounded

A measure ω_Λ is called Dirac comb.

It's called δ_Λ when $\omega \equiv 1$, and called a weighted Dirac comb if not.

Ex. 9.1 (P336)

Let Λ be a point set of FLC.

Consider the Dirac comb ω_Λ with bounded ω

Assume its natural autocorrelation γ_ω exists.

$\Lambda_R := \Lambda \cap \overline{B_R}$, $\widetilde{\omega_\Lambda} = \sum_{x \in \Lambda} \overline{\omega(x)} \delta_{-x}$

$\omega_{\Lambda R} * \widetilde{\omega_{\Lambda R}} = \left(\sum_{x \in \Lambda R} \omega(x) \delta_x\right) * \left(\sum_{y \in \Lambda R} \overline{\omega(y)} \delta_y\right)$, let $\phi \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}
\omega_{\Lambda_R} * \widetilde{\omega_{\Lambda_R}}(\phi) &= \iint \phi(s+t)\, d\omega_{\Lambda_R}(s)\, d\widetilde{\omega_{\Lambda_R}}(t) \\
&= \int \sum_{x \in \Lambda_R} \omega(x) \phi(x+t)\, d\widetilde{\omega_{\Lambda_R}}(t) \\
&= \sum_{x \in \Lambda_R} \omega(x) \sum_{y \in \Lambda_R} \overline{\omega(y)} \phi(x-y), \text{ let } z = x-y \\
&= \sum_{x \in \Lambda_R} \sum_{x-z \in \Lambda_R} \omega(x) \overline{\omega(x-z)} \phi(z)
\end{aligned}$$

$$\omega_{\Lambda_R} * \widetilde{\omega_{\Lambda_R}} = \sum_{x \in \Lambda_R} \sum_{x-z \in \Lambda_R} \omega(x) \overline{\omega(x-z)} \delta_z$$

$$\gamma_\omega = \lim_{R\to\infty} \frac{1}{\mathrm{vol}(B_R)} \sum_{x\in\Lambda_R} \sum_{x-z\in\Lambda_R} \omega(x)\overline{\omega(x-z)}\,\delta_z$$

$$= \lim_{R\to\infty} \sum_{z\in\Lambda_R-\Lambda_R} \underbrace{\frac{1}{\mathrm{vol}(B_R)} \sum_{\substack{x\in\Lambda_R \\ x-z\in\Lambda_R}} \omega(x)\overline{\omega(x-z)}}_{=:\,\eta_R(z)}\,\delta_z$$

$\therefore \gamma_\omega$ exists iff $\lim\limits_{R\to\infty} \eta_R(z) =: \eta(z)$ exists pointwise $\forall\, z\in\Lambda-\Lambda$

Then $\gamma_\omega = \cancel{\sum_R} \sum\limits_{z\in\Lambda-\Lambda} \eta(z)\,\delta_z$