Reminder

Let $\Gamma \subset \mathbb{R}^d$ a lattice (ex. $\mathbb{Z}^d$)

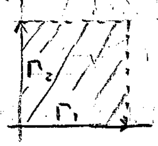
A function is Γ-periodic if (for $f : \mathbb{R}^d \to \mathbb{R}$)

$$\underset{\delta_x * f}{\underset{\parallel}{f(\cdot - x)}} = f(\cdot) \ \forall x \in \Gamma$$

A measure $\mu \in M(\mathbb{R}^d)$ is Γ-peroidic if

$$\delta_x * \mu = \mu \ \forall x \in \Gamma$$

$FD(\Gamma) = [0, |\Gamma_1|)[0, |\Gamma_2|)$



Proposition

Let $FD(\Gamma)$ (:= fundamental domain of Γ)

If $\mu \in M(\mathbb{R}^d)$ is Γ-periodic then

$$\exists \rho \in M(FD(\Gamma)) : \mu = \rho * \delta_\Gamma \equiv \rho * \Big(\sum_{x \in \Gamma} \delta_x\Big)$$

Proof

We set $\rho = \mu|_{FD(\Gamma)}$ and observe that

$$\mu|_{FD(\Gamma)+x} = \delta_x * \rho, \ \forall x \in \Gamma$$

$$\Rightarrow \mu = \sum_{x \in \Gamma} \delta_x * \rho = \rho * \delta_\Gamma$$

$\square$

Then

$$\gamma_\Gamma = \lim_{R \to \infty} \frac{1}{\mathrm{vol}(B_R)} \delta_\Gamma|_{B_R} * \widetilde{\delta_\Gamma|_{B_R}} = \mathrm{dens}(\Gamma)\, \delta_\Gamma$$

and $\hat{\gamma}_\Gamma = \mathrm{dens}^2(\Gamma) \overset{\delta}{\cancel{\gamma}}_{\Gamma^*}$ (by 井上さん)

If $\mu = \rho * \delta_\Gamma$, then

$$\gamma_\mu = \rho * \tilde{\rho} * \delta_\Gamma = \mathrm{dens}(\Gamma)\, \rho * \tilde{\rho} * \delta_{\Gamma^*}$$

and $\hat{\gamma}_\mu = \mathrm{dens}^2(\Gamma) |\hat{\rho}|^2 \delta_{\Gamma^*}$

$$\hat{\gamma}_\mu(\{0\}) = \mathrm{dens}^2(\Gamma) |\hat{\rho}(0)|^2$$