muchammad yasin - range OCT 12 2017

Calculus 1 week 2

$|a-b| \leq \varepsilon \Leftrightarrow -\varepsilon \leq a-b \leq \varepsilon \Leftrightarrow b-\varepsilon \leq a \leq b+\varepsilon$

I.3 Function

☆ Recall that a set is a collection of objects such as $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, (a,b), {G30 students}

☆ Def: a function ($\equiv$ a map) from a set X to set Y is a rule which associates a unique element of Y to each element of X.

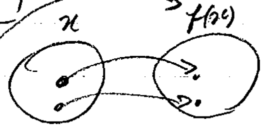
☆ We call the domain of 'f' denote it by $Dom(f)$,

☆ Y is the codomain of 'f', or the target space.

☆ $Ran(f) = f(x) = Im(f)$ (range)
$= \{y \in Y \mid \exists x \in X \text{ with } y = f(x)\}$

Notation: We use
$f: X \ni x \mapsto f(x) \in Y$
OR $f: X \longrightarrow$
$x \longmapsto f(x)$

⚠ "$f(x)$" is not a function
↳ it is a value



Def: Let $f: X \to Y$

- f is injective if for any $x_1, x_2 \in X$, $x_1 \neq x_2$, one has $f(x_1) \neq f(x_2)$
- f is surjective if for any $y \in Y$, there exists an $x \in X$ such that $y = f(x)$
- bijective → both.

Example: $f: \mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}$ not inj., not surj.
$f: \mathbb{R}_+ \ni x \mapsto x^2 \in \mathbb{R}$ inj., not surj.
$f: \mathbb{R}_+ \ni x \mapsto x^2 \in \mathbb{R}_+$ bijective
$f: \mathbb{R} \ni x \mapsto x^2 \in \mathbb{R}_+$ not inj., surj.

I.4 Operation on function

(can be defined if Y is a vector space)

☆ Addition: Let $f,g: X \to Y$; what is $f+g$?
Example: $f,g: \mathbb{R} \to \mathbb{R}$, $(f+g)(x) = f(x) + g(x)$

can be defined if Y is an algebra

☆ Multiplication: Let $f,g: X \to Y$; ~~What is~~ fg?
— : $f,g: \mathbb{R} \to \mathbb{R}$, then $(fg)(x) = f(x)\,g(x)$

☆ Composition: Let $f: X \to Y$ and $g: Y \to Z$
set $g \circ f \Leftrightarrow (g \circ f)(x) = g(f(x))$

only if g is defined on f's set.

● Well defined if $Ran(f) \subset Y' \equiv Dom(g)$
range of 'f' is inside domain of 'g'

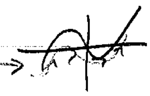
Peano Curve!

II Graphs and Curves

II.1 Graphs

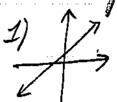
Consider $f: \mathbb{R} \to \mathbb{R}$, we represent it by graphic image

The set $\{(x, f(x)) \in \mathbb{R}^2 \mid x \in \mathbb{R}\}$ is the graph of the function.



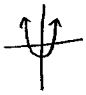
examples

1)



straight line

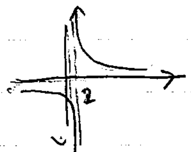
2)



parabola

$f: \mathbb{R} \to \mathbb{R}$

3)



$\boxed{f: \mathbb{R}/\{2\} \to \mathbb{R}}$ ☆

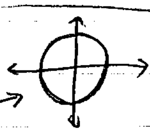
↳ this one means it excludes '2'.

A parametric curve is a function $f: I \to \mathbb{R}^2$ with I an interval of $\mathbb{R}$

$t \longmapsto (x(t), y(t))$

II.2 Curves

1) $\{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ →

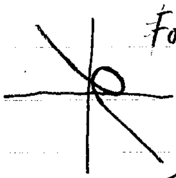


2) $\{(x,y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$

3) $\{(x,y) \in \mathbb{R}^2 \mid x = \frac{3t}{1+t^3},\ y = \frac{3t^2}{1+t^3} \text{ for } t \in \mathbb{R}\}$

↳ This is a parametric curve.

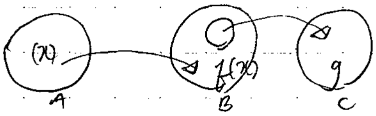
Folium of Descartes.



pre-college Week 2

OCT. 11. 2017

a function f from A to B is a rule which assignes an element of B to any value $x \in \mathbb{R}$



$g \circ f$

the definition of well defined where the values of g is properly restricted.

$f : \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$
$g : \mathbb{R} \to \mathbb{R}$

follows the one inside

a) $f \circ g = f(g(x))$
$= (\sqrt{x} - 1)^2$

is this meaningful/correct for $x \in \mathbb{R}_+$

f is injective if for any $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$

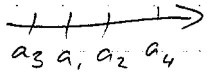
f is surjective if for any $y \in B$, there exists $x \in A$ with $f(x) = y$

f is bijective if it's both injective & surjective.

SEQUENCE

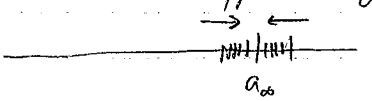
$(a_n)_{n \in \mathbb{N}}$ called a sequence

Question: does this sequence converge to something!?



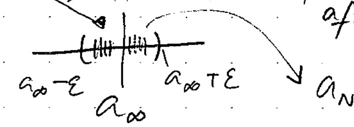
Def: the sequence converges to a_∞ if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ large enough such that $|a_n - a_\infty| \leq \varepsilon$ $\forall n \geq N$.

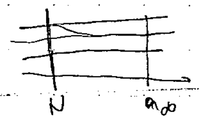
approaches a_∞



approaches, within frame of ε

after a certain N,

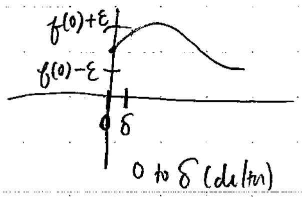




FUNCTION

$f : (0, \infty) \to \mathbb{R}$ What means of f has a limit at 0?

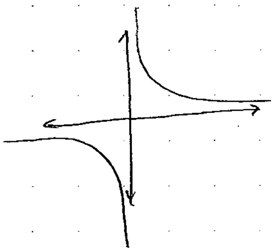
f has a limit at $x = 0$ if there exists a number (which we denote by $f(0)$ [such that for any $\varepsilon > 0$, $\forall \delta > 0$ such that $|f(x) - f(0)| \leq \varepsilon$ for any $|x - 0| < \delta$]
$x < \delta$



0 to δ (delta)

in this case we write $\lim_{x \to 0} f(x) = f(0)$

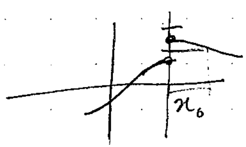
If I choose $f(x) = 25$ for this graph $f(x) = \frac{1}{x}$



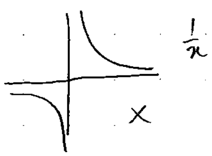
FIND THE LOGIC OF EPSILON AND DELTA HERE.
↓

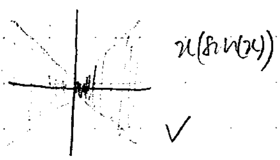
☆ If f has a limit at 0, we write $\lim_{x \to 0} f(x) = f(0)$

Def: $f: \mathbb{R} \to \mathbb{R}$ is continuous at x_0 if $\forall \varepsilon > 0$, $\forall \delta > 0$ such that $|f(x) - f(x_0)| \leq \varepsilon$ for any $x \in (x_0 - \delta, x_0 + \delta)$



always consider the tube/space between





no limit at 0

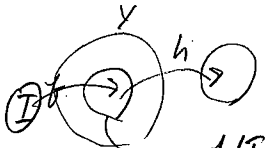
Remark:
f is continuous at x_0 if and only if
$$\lim_{\substack{x \to x_0 \\ x > x_0}} f(x) = f(x_0)$$
$$\lim_{\substack{x \to x_0 \\ x < x_0}} f(x) = f(x_0)$$

Def: Let $f: (a, b) \mapsto \mathbb{R}$ is continuous on (a, b) if it is continuous at any $x_0 \in (a, b)$.
The set of all continuous on (a, b) is denoted by $C((a,b))$.

Calculus week 3 OCT 19 2017

f is continuous on interval I, if f is continuous at any $x_0 \in I$,
we write $f \in C(I)$.

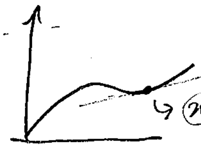
Lemma
Let I be an interval and let $f, g \in C(I)$, (a,b)
Let $\lambda \in \mathbb{R}$
1) $\lambda f + g \in C(I)$
2) $fg \in C(I)$
3) $f/g \in C(I)$ if $g(x) \neq 0$ for any $x \in (I)$



$f(I) = \operatorname{Img}(f) = \operatorname{Ran} f$

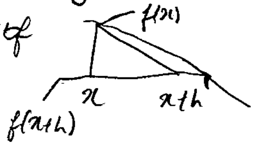
☆ If $f \in C(I)$ and let $h \in C(f(I))$ ← h is continuous on $f(I)$
one has $h \circ f \in C(I)$

III. 2 Slope of a curve at a point
Idea

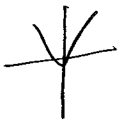
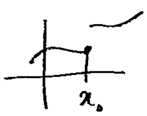


The slope of a straight line tangent to the curve at x is called the slope of the curve at x.

more precisely: let $f: I \to \mathbb{R}$ be a function describing the curve locally, and let $x \in I$ and $x+h \in I$ for h small enough.
The $(x, f(x)) \in \mathbb{R}^2$ and $(x+h, f(x+h)) \in \mathbb{R}^2$
☆ ☆
belong to the curve and we can consider the slope of



☆ Definition ⟶ the slope of f at x is defined by the limit below:
$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Example: 1)  2)  3) 

III.3 The Derivative of a function

Let: $f : I \to \mathbb{R}$ and let $x \in I$,

Def: the value $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ (if it exists) is called the derivative of 'f' at x, and denoted by $f'(x)$ or $\frac{df}{dx}(x)$

If $f'(x)$ exists for any $x \in I$ we consider the function
$f'(x) : I \longrightarrow \mathbb{R}$, $f(x) := f'(x)$

we write this be cause it can be obtained on every interval of f to find its slope.

Examples 1) $f : \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$

Then: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \frac{(x+h)^2 - x}{h}$

$= \lim_{h \to 0} \frac{2xh + h^2}{h}$

$= \lim_{h \to 0} 2x + h$

$= 2x$

☆ the derivative of f is not '$2x$', it's the function $f' : \mathbb{R} \to \mathbb{R}$,
$f'(x) = 2x$

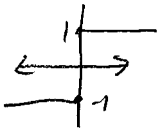
☆ or the derivative is
$\mathbb{R} \ni x \mapsto 2x \in \mathbb{R}$

Example 2) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$

for $x > 0$, one has $\frac{f(x+h) + - f(x)}{h} = 1$ for $x + h > 0$

for $x < 0$, one has $\frac{f(x+h) - f(x)}{h} = \frac{-(x+h) + x}{h}$ for $x + h < 0$
$= -1$

If $x = 0$, then $\frac{f(x+h) - f(x)}{h} = \frac{|h| - 0}{h} = \begin{cases} 1 & \text{if } h > 0 \\ -1 & \text{if } h < 0 \end{cases}$



↳ it has no limit when $h \to 0$
↳ the derivatives of f at 0 does not exist.

Remark

☆ In 1D (what we do in Calc 1) the derivatives of f at x and the slope of the curve defined by f are the same thing! (not in higher dimensions).

☆ We could define $f'_+(x) = \lim_{\substack{h \to 0 \\ h > 0}} \frac{f(x+h) - f(x)}{h}$; $f'_-(x) = \lim_{\substack{h \to 0 \\ h < 0}} \frac{f(x+h) - f(x)}{h}$
and these limits could be different, like with $f(x) = |x|$.

Remark

When we write $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$ for any $\forall$ $\varepsilon > 0$, $\exists\, \delta > 0$ such that $\left| \frac{f(x+h)-f(x)}{h} - f'(x) \right| \le \varepsilon$ for any $h \in (-\delta, \delta)$.

Remark If f is differentiable on I, its derivative is the function $f' : I \to \mathbb{R}$

If $f' \in C(I)$, we write $f \in C^1(I)$ — meaning f has a derivative and its derivative is continuous.

III.4 Back to limits

$\mathbb{R}^*$: all value except zero

Let $f, g : \mathbb{R}^* \to \mathbb{R}$ be continuous at $\mathbb{R}^*$ and assume that $\lim_{x\to 0} f(x)$ exists & $\lim_{x\to 0} g(x)$ exists,

→ and we call it $f(0)$ → we call it $g(0)$.

behind these, there is ε

Lemma:

- $\lim_{x\to 0} (f(x) + g(x)) = f(0) + g(0)$.
- $\lim_{x\to 0} f(x)\cdot g(x) = f(0)\, g(0)$
- $\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{f(0)}{g(0)}$ if $g(0) \neq 0$

If $f(0) = 0 = g(0)$, what is $\lim_{x\to 0} \frac{f(x)}{g(x)}$?

examples 1) $f(x) = x^2$ $g(x) = x^2 + x$

one has $\frac{f(x)}{g(x)} = \frac{x^2}{x^2+x} = \frac{1}{1+x} \underset{x\to 0}{\Longrightarrow} 0$

2) $f(x) = x^2$ $g(x) = x^3$

$\frac{f(x)}{g(x)} = \frac{x^2}{x^3} = \frac{1}{x} \Rightarrow$ undefined limit when $x \to 0$

3) $f(x) = x^2$, $g(x) = x^2 + x^3$

$\frac{f(x)}{g(x)} = \frac{x^2}{x^2+x^3} = \frac{1}{1+x} \underset{x\to 0}{\Longrightarrow} 0$

Theorem l'Hopital's Rule

Let $f, g : I \to \mathbb{R}$ and let $x_0 \in I$. Assume that $f(x_0) = 0 = g(x_0)$ and $g(x) \neq 0$ for $x \neq x_0$. Assume that f, g are differentiable at I.

this covers all, both $x \cup x_0$

to ensure its validity to be ~~dif~~ differentiated at $x \cup x_0$

Then the limit $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ if this limit exists.

Proof: One has

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \lim_{x\to x_0} \frac{\frac{f(x)-f(x_0)}{x-x_0}}{\frac{g(x)-g(x_0)}{x-x_0}} = \frac{\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}}{\lim_{x\to x_0} \frac{g(x)-g(x_0)}{x-x_0}}$$

$$= \frac{f'(x_0)}{g'(x_0)} =$$

Example.

$f(x) = x^2$, $g(x) = x^2 + x^3$ $\quad \lim_{x\to 0} \frac{x^2}{x^2+x^3}$ apply L'Hopital's Rule.

continuation on next page.

III.5 Properties of differentiation

Proposition: Let $f, g: I \to \mathbb{R}$ be two differentiable functions on I and let $x \in \mathbb{R}$. ~~Then~~ let $\lambda \in \mathbb{R}$.

1) $(\lambda f + g)' = \lambda f' + g'$
2) $(fg)' = f'g + fg'$
3) $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$

③ do your own proving.

Proof

① $= \frac{(\lambda f+g)(x+h) - (\lambda f+g)(x)}{h}$

$= \frac{\lambda f(x+h) + g(x+h) - \lambda f(x) - g(x)}{h}$

$= \lambda \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}$

$= \lim_{h\to 0} ☆ = \lambda f'(x) + g'(x)$

② $= \frac{(fg)(x+h) - (fg)(x)}{h} = \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$

$= \frac{f(x+h)(g(x+h)-g(x)) + g(x)(f(x+h)-f(x))}{h}$

$= f(x+h) \frac{(g(x+h)-g(x))}{h} + g(x) \frac{(f(x+h)-f(x))}{h}$

$= f(x+h)\, g'(x) + g(x)(f'(x))$

$\xrightarrow{h\to 0} f(x)g'(x) + g(x)f'(x)$

Again. L'Hôpital's Rule

Let $f, g : I \to \mathbb{R}$ be differentiable. $x_0 \in I$. Assume that $f(x_0) = 0 = g(x_0)$, $g(x) \neq 0$ for $x \neq x_0$, $g'(x) \neq 0$ for $x \neq x_0$

Then $\lim_{x \to x_0} f(x)/g(x) = \lim_{x \to x_0} f'(x)/g'(x)$ if the limit exist

$\boxed{\text{Proof (if } g'(x_0) \neq 0)}$ on prev page.

$\lim_{x \to x_0} \frac{f(x)}{g(x)}$ THIS PART IS JUST A REPETITION

$(f \circ g)'$ Lemma Chain Rule

Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable.

Then $(f \circ g)'(x) = f'(g(x))\, g'(x)$

- Proof

one has $\frac{f(g(x+h)) - f(g(x))}{h}$

let $u := g(x)$

$k = g(x+h) - g(x)$

☆ observe that $k \to 0$ when $h \to 0$

$= \frac{f(u+k) - f(u)}{h}$

$= \frac{f(u+k) - f(u)}{k} \frac{k}{h}$

$= \frac{f(u+k) - f(u)}{k} \frac{g(x+h) - g(x)}{h}$

$= f'(u)\, g'(x)$

$= f'(g(x))\, g'(x)$

II.6 Higher Derivatives

Consider $f : I \to \mathbb{R}$ be differentiable $\Rightarrow f' : I \to \mathbb{R}$. If f' is differentiable,

- we can consider $(f')' = f'' = f^{(2)} \neq f^2$
- can be iterated $f^{(n)} = f^{'''\ldots'}$
= if f' is continuous, we write $f \in C^1(I)$ ← first derivative of f continuous.
- If f is continuous, we write $f \in C^n(I)$

Example: let $P_n(x) = x^n$

$P_n'(x) = nx^{n-1}$

$P_n''(x) = n(n-1)x^{n-2}$

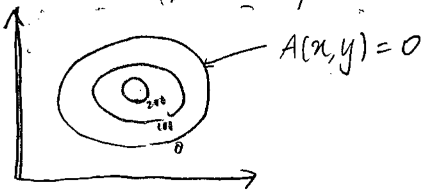
$\vdots$

$P_n^{(n)}(x) = n!$ ← constant.

$P_n^{(n+1)}(x) = 0$

II.7 Implicit Differentiation

Quite often, a curve in $\mathbb{R}^2$ can be described by a relation of the form $F(x,y)=0$ for $F: \mathbb{R}^2 \to \mathbb{R}$



☆ Aim: if (x,y) satisfies $F(x,y)=0$ what is the slope of the tangent of the curve at (x,y).

↳ example: consider the given curve:

$$F(x,y) = 3x^2y - y^4 + 5x^2 + 5 = 0$$

keyword: solve/find the tangent locally, by means a restricted area.

If we could solve $y = \ldots$ as a function of x, then we can just differentiate as a function

▼ Assume that locally, we can produce the equation $y = y(x)$.

Then $\frac{dF}{dx}(x,y) = 6xy + 3x^2\frac{dy}{dx} - 4y^3\frac{dy}{dx} + 10x = 0$

$$(3x^2 - 4y^3)\frac{dy}{dx} = -6xy - 10x$$

$$\frac{dy}{dx} = \frac{-6xy - 10x}{3x^2 - 4y^3}$$

we have both x and y in the derivative because we are to find the slope 「LOCALLY」.

▼ Consider $(x,y) = (1,2)$

check that $F(1,2) = 6 - 16 + 5 + 5 = 0$

then $\frac{dy}{dx}(1,2) = \frac{-10-12}{3-32} = \frac{22}{29}$

! If $f: I \to \mathbb{R}$ is differentiable, then f is continuous

Indeed for $\varepsilon > 0$, $\exists\, \delta > 0$ such that $\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| \le \varepsilon$

$\forall |h| \le \delta$ $\Updownarrow$

$$-\varepsilon \le \frac{f(x+h) - f(x)}{h} - f'(x) \le \varepsilon$$

$$-\varepsilon + f'(x) \le \frac{f(x+h) - f(x)}{h} \le \varepsilon + f'(x)$$

$$h(-\varepsilon + f'(x)) \le f(x+h) - f(x) \le h(\varepsilon + f'(x))$$

$\boxed{\text{if } h > 0}$

Corallary

IV Some basic function

IV.1 Sine and cosine function

$\sin, \cos : \mathbb{R} \to [-1, 1]$, 2π period

★ some trigonometric identities

$\cos^2(x) + \sin^2(x) = 1$

$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$

$\cos(x+y) = \cos(x)\cos(y) + -\sin(x)\sin(y)$

$\sin(x) - \sin(y) = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$

$\cos(x) - \cos(y) = -2\sin\left(\frac{x-y}{2}\right)\sin\left(\frac{x+y}{2}\right)$

✿ Lemma → $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$. Corollary : $\sin'(x) = \cos(x)$

Proof

$$\sin'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{2\sin\left(\frac{h}{2}\right)\cos\left(\frac{2x+h}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \cdot \cos\left(\frac{2x+h}{2}\right)$$

$$= \lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{h/2} \lim_{h\to 0} \cos\left(\frac{2x+h}{2}\right)$$

$$= 1 \cdot \cos x = \cos x$$

Corollary

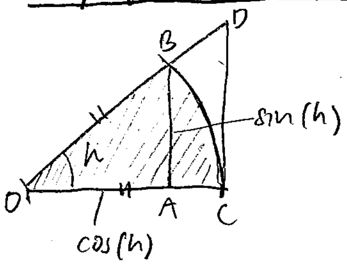
1) $\cos'(x) = -\sin(x)$

2) $\lim_{x\to 0} \frac{\cos(x) - 1}{x} = 0$

3) $\lim_{x\to} \frac{\cos(x) - 1}{x^2} = -\frac{1}{2}$

proved in HS

Proof of the lemma ✿



$\tan(h) = \frac{\sin(h)}{\cos(h)}$

- h is the angle of the radian
- consider ΔOAB and ΔOCD and the portion of the disk (shaded part)

$\Delta OAB \leq$ shaded part $\leq \Delta OCD$

$$\frac{1}{2}\cos(h)\sin(h) \leq \frac{h}{2\pi}\cdot\pi \leq \frac{1}{2}\frac{\sin(h)}{\cos(h)} \Leftrightarrow$$

$$\cos(h)\sin(h) \leq h \leq \sin(h)/\cos(h) \Leftrightarrow$$

$$\cos(h) \leq h/\sin(h) \leq 1/\cos(h) \Leftrightarrow$$

$$\frac{1}{\cos(h)} \geq \frac{\sin(h)}{h} \geq \cos(h) \Leftrightarrow$$

since $\lim_{h\to 0} \cos(h) = 1$ and $\lim_{h\to 0} \frac{1}{\cos(h)} = 1$

$$1 = \lim_{h\to 0} \frac{1}{\cos(h)} \geq \frac{\sin(h)}{h} \geq \lim_{h\to 0} \cos h = 1$$

IV.2 The exponential function

We have seen that $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ is well defined, and $f'(x) = f(x)$

We call it exponential function and write

$f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ $\qquad e^0 = 1$

just a notation $\qquad e^1 = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.71\ldots$

Lemma Let $g: \mathbb{R} \to \mathbb{R}$ b differentiable and such that $g(x) = g'(x)$
Then $\underline{g(x) = ce^x}$ for some $c \in \mathbb{R}$ and all $x \in \mathbb{R}$

Proof let us show that $x \mapsto \frac{g(x)}{e^x}$ is a constant (so its derivatives must be zero $\emptyset$)

indeed $\frac{d}{dx}\left(\frac{g(x)}{e^x}\right) = \frac{g'(x)e^x - (e^x)'g(x)}{(e^x)^2}$

$= \frac{g(x)e^x - e^x g(x)}{(e^x)^2} = 0 \quad \forall x \in \mathbb{R}$

⚠ it must also be shown that $e^x \neq 0$

$\Rightarrow \frac{g(x)}{e^x} = c \Leftrightarrow g(x) = ce^x$

V The Mean Value Theorem

V.1 Local Maximum & Local Minimum : Consider $f: I \to \mathbb{R}$

definition: → a point $x_0 \in I$ is a local maximum on I if $\exists \delta > 0$ such that
$f(x_0 + h) \leq f(x_0)$ for all $|h| \leq \delta$

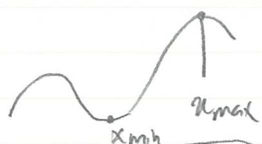
→ a point $x_0 \in I$ is a local minimum on I if $\exists \delta > 0$ such that
$f(x_0 + h) \geq f(x_0)$ for all $|h| \leq \delta$

⚠ local max or min donot always exist

Theorem Extreme Value Theorem

Let $f: [a,b] \to \mathbb{R}$ be continuous, then there exist
$x_{min} \in [a,b]$ and $x_{max} \in [a,b]$ such that

$f(x_{max}) \geq f(x)$ for any $x \in [a,b]$
$f(x_{min}) \leq f(x)$ for any $x \in [a,b]$



⚠ → x_{max} & x_{min} are not always unique
↳ this only applies on closed interval!

hello

We say that a local maximum or minimum x_0 on I is
a global maximum → if $f(x_0) \geq f(x) \quad \forall x \in I$
or minimum → if $f(x_0) \leq f(x) \quad \forall x \in I$

→ applies to all/entire interval of I (not closed)

☆ A related result : **Theorem** : Intermediate Value Theorem

Let $f: [a,b] \to \mathbb{R}$ be continuous and set $m := f(x_{min})$, $M := f(x_{max})$,
Then for any $y \in [m, M]$ there exist at least one $x \in [a,b]$ with $f(x) = y$

continued →

Def: Let $f: I \to \mathbb{R}$ be differentiable. A point $x_0 \in I$ is critical ($\equiv$ a critical point for f) if $f'(x_0) = 0$

Examples: 1) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \cos(x)$ then $f'(x) = -\sin(x)$ and the critical points are $\{n\pi \text{ with } n \in \mathbb{Z}\}$

2) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = (x-1)^2$ then $f'(x) = 2(x-1)$

critical point $= \{1\}$

3) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ then $f'(x) = 3x^2$

critical point $= \{0\}$

☆ **Theorem:** Let $f: I \to \mathbb{R}$ be differentiable and let $x_0 \in I$ be a local maximum or local minimum if $f'(x_0) = 0$

! the converse is not always true

Proof: Assume x_0 is a local maximum, then for $h > 0$ and small enough, one has

$$\frac{f(x_0+h) - f(x_0)}{h} = -\frac{\overbrace{f(x_0) - f(x_0+h)}^{\geq 0}}{\underbrace{h}_{>0}} \leq 0$$

for $h < 0$ and small enough $\rightarrow$

$$\frac{f(x_0+h) - f(x)}{h} \geq 0$$

⊛ Since f is differentiable,

$$\underbrace{\lim_{h \to 0+} \frac{f(x_0+h) - f(x_0)}{h}}_{\leq 0} = \underbrace{\lim_{h \to 0-} \frac{f(x_0+h) - f(x_0)}{h}}_{\geq 0} = 0$$

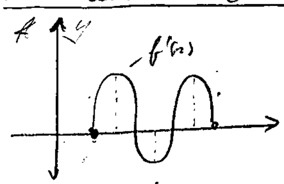
(only value which both ≥ 0 & ≤ 0)

It implies

$$\lim_{h \to 0} \frac{f(x_0+h) + - f(x)}{h} = 0$$

$\frac{x(x+1)}{x-1}$

V.2 Mean Value Theorem



There exist at least one $f'(x) = 0$

☆ **Theorem** (Rolle's theorem).

→ Let $f: [a,b]$ (closed interval) be continuous are differentiable at (a,b) (open interval),

→ Assume $f(a) = f(b) = 0$, then there exists $x_0 \in (a,b)$ such that $f'(x_0) = 0$

Special case to back up the theorem:

$y = \sqrt{x}$ at 0

$(y)' \neq 0$

☆ **Proof**

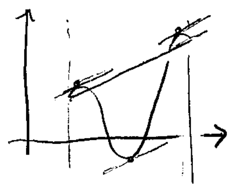
→ if $f = 0$ then for any $x_0 \in (a,b)$ one has $f'(x_0) = 0$

→ if $f \neq 0$ there exists $x \in (a,b)$ with $f(x) \neq 0$

→ Assume $f(x) \gtrless 0$, then by the extreme value theorem, there exists a x_{max} (x_{min}) such that $f(x_{max}) \geq f(x)$ ($f(x_{min}) \leq f(x)$)

By the last theorem, one has $f'(x_{max}) = 0$

$x_{min} = 0$



☆ Theorem - Mean Value Theorem

→ Let $f: [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b)

Then there exists $c \in (a,b)$ with $f'(c) = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{slope of the green line.}}$

☆ Proof

→ Consider the line of the equation $y = \frac{f(b)-f(a)}{b-a} \cdot (x-a) + f(a)$

→ Consider $h: [a,b] \to \mathbb{R}$ given by:

$$h(x) = f(x) - \left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a) \right]$$

this one is to find the difference between the function & the slope of two points a & b.

☆ Observe that: $h(a) = 0 = h(b)$

By Rolle's theorem, $\exists c \in (a,b)$ s.t. $h'(c) = 0$.

But $0 = h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

☆ Corollary

Let $f: I \to \mathbb{R}$ be differentiable and such that $f'(x) = 0 \ \forall x \in I$, then f = constant.

☆ Proof

→ Consider $[a,b] \subseteq I$ with $a<b$. (symbol)

Then $f: [a,b] \to \mathbb{R}$ is continuous and differentiable on (a,b).

→ By the previous theorem, $\exists c \in (a,b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$. (see)

Since $f'(c) = 0$, one gets $f(a) = f(b)$

→ Since a and b are arbitrary points in I, one infers that $f(x)$ = constant $\forall x \in I$

☆ New Proof of l'Hôpital's Rule

→ Step 1 Cauchy mean value theorem.

Let $f, g: [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b)

Then $\exists c \in (a,b)$ such that $f'(c)(g(b)-g(a)) = g'(c)(f(b)-f(a))$ ⊛

☆ Proof

1) if $g(b) = g(a)$, $\exists c \in (a,b)$ with $g'(c) = 0$ by Rolle's theorem (shifted)

$\Rightarrow$ ⊛

2) if $g(b) \neq g(a)$, set $h: [a,b] \to \mathbb{R}$ with $h(x) := f(x) - \frac{f(b)-f(a)}{g(b)-g(a)} g(x)$

then $h(a) = \frac{f(a)g(b) - f(b)g(a)}{g(b)-g(a)} = h(b)$ $\Rightarrow$ by Rolle's Theorem (shifted) there exists $c \in (a,b)$ with $h'(c) = 0$

$\Rightarrow f'(c) - \frac{f(b)-f(a)}{g(b)-g(a)} g'(c) = 0$ □

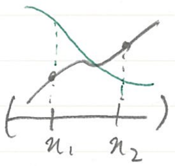
continued →

Step 2 Proof of l'Hopital's Rule

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(c)}{g'(c)} \quad \text{for some } c \in (x, x_0)$$

Taking the limit $x \to x_0$ implies that $c \to x_0$ $\Rightarrow \lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \lim\limits_{c \to x_0} \frac{f'(c)}{g'(c)}$

Definition: A function $f: I \to \mathbb{R}$ is increasing if $f(x_1) \overset{\geq}{\leq} f(x_2)$ for any $x_1, x_2 \in I$ and $x_1 \leq x_2$. It is strictly increasing if $f(x_1) \underset{>}{<} f(x_2)$ when $x_1 < x_2$



eg: differentiable but not continuous
↳ $1/x$

these two are important

Lemma: Let $f: [a,b] \to \mathbb{R}$ be <u>continuous</u> and <u>differentiable</u> on (a,b)
if $f'(x) \geq 0$ for any $x \in (a,b)$, then f is increasing on (a,b)
if $f'(x) \leq 0$ for any $x \in (a,b)$, then f is decreasing —"—

Proof Choose $x_1, x_2 \in (a,b)$ with $x_1 < x_2$. Then by the mean value theorem, $\exists\, c \in (x_1, x_2)$ such that

first case
$$0 \leq f'(c) = \frac{f(x_2)-f(x_1)}{x_2 - x_1} \Rightarrow f(x_2) \geq f(x_1)$$
second case
$$0 \geq f'(c) = \text{—"—} \Rightarrow f(x_2) \leq f(x_1)$$

Since x_1, x_2 are arbitrary, we get the statement. □

☆ Same statement with strict inequality and strictly increasing or decreasing.

No 3 Exponential and Logarithmic functions

Remember, we set $f(x) := \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ and checked that $f'(x) = f(x)$ & $f(0) = 1$

$(f(-x))' = (-x)' f'(-x)$
$= -f'(x) = -f(-x)$

Lemma
1) $f(x) f(-x) = 1 \quad \forall x \in \mathbb{R}$
2) $f(x) > 0 \quad \forall x \in \mathbb{R}$
3) f is strictly increasing
4) $f(x+y) = f(x) f(y) \quad \forall x, y \in \mathbb{R}$

Proof
1) consider $(f(x) f(-x))' = f'(x) f(-x) + f(x)(f(-x))'$
$= f(x) f(-x) + f(x)(-f'(-x))$
$= f(x) f(-x) - f(x)(f(-x))$
$= 0$

$\Rightarrow$ by previous lemma, it follows that $f(x) f(-x) = $ constant
by choosing $x = 0$, one has $f(0) f(-0) = f(0)^2 = 1$

⚠ proof 2 is not applicable? proof 1 is true

2) By 1) $f(x) \neq 0$ for any $x \in \mathbb{R}$ otherwise $1 = f(x) f(-x) = 0$ $f(-x) = 0$
By the intermediate value theorem, one must have either $f(x) > 0$ $\forall x \in \mathbb{R}$ or $f(x) < 0$ for all $x \in \mathbb{R}$
Since $f(0) = 1 > 0 \Rightarrow f(x) > 0$ $\forall x \in \mathbb{R}$

2) Since $f'(x) = f(x) > 0$

$\Rightarrow$ by the previous lemma that f is strictly increasing

4) For fixed $x \in \mathbb{R}$, consider the function

$$\phi : \mathbb{R} \to \mathbb{R}, \quad \phi(Y) = \frac{f(x+Y)}{f(Y)}$$

then $$\phi'(Y) = \frac{f'(x+Y)f(Y) - f'(Y)f(x+Y)}{f(Y)^2} = 0$$

this implies that $\Rightarrow \phi(Y) = \text{constant} = \phi(0)$

we get $$\phi(Y) = \frac{f(x+Y)}{f(Y)} = \frac{f(x)}{f(0)} = f(x).$$

Conclusion: We are going to call this function the exponential function, and denote it by $e^x \equiv f(x)$

just a notation

$$\Rightarrow \begin{cases} e^x e^{-x} = 1 \\ e^x > 0 \\ x \mapsto e^x \text{ is increasing} \\ e^{x+y} = e^x e^y \end{cases}$$

Summary of Homework

The function $\mathbb{R} \ni x \mapsto x^x \in \mathbb{R}_+ \equiv (0,\infty)$ satisfies,

1) $(e^x)' = e^x$
2) The function is increasing strictly
3) $e^{-x} = 1/e^x$
4) $e^{y+x} = e^x e^y$

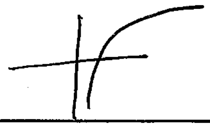
I'm ~~late~~ 2 mins late to this class and this whole thing is already on board!

Its increase is the logarithmic function $(0,\infty) \ni y \mapsto \ln(y) \in \mathbb{R}$ which satisfies

1) $e^{\ln(y)} = y$ and $\ln(e^x)\ x$
2) $(\ln(y))' = \frac{1}{y}$
3) the function is strictly increasing
4) $\ln(ab) = \ln(a) + \ln(b)$
5) $\ln(y^x) = x \ln(y)$ for any $y \in \mathbb{R}_+$, $x \in \mathbb{R}$

Then we set

$$y^x := e^{x \ln(y)} \quad \text{for any } y \in \mathbb{R}_+,\ x \in \mathbb{R}$$



VI Inverse Function

Question: when can one invert a function?

<u>Theorem</u> (proven in homework).

Let $f : [a,b] \to \mathbb{R}$ b strictly increasing and set $\alpha = f(a)$, $\beta = f(b)$

Then there exist $f^{-1} : [\alpha, \beta] \to [a,b]$ such that:

$$f^{-1} \circ f(x) = x \quad \& \quad f \circ f^{-1}(y) = y$$
$$\forall x \in [a,b],\ y \in [\alpha, \beta]$$

Remark: A similar statement holds if f is strictly decreasing.

<u>Theorem</u> Let $f : [a,b] \to \mathbb{R}$ be a continuous function and differentiable on (a,b) with $f'(x) > 0$ for any $x \in (a,b)$.

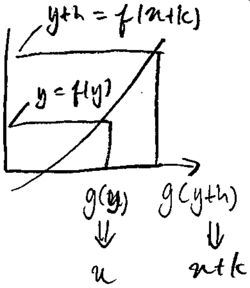
Then, by setting $\alpha := f(a)$, $\beta := f(b)$, the <u>inverse function</u> $f^{-1} : [\alpha,\beta] \to [a,b]$ is differentiable on (α,β) with

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}$$ ← use y because it's the inverse.

<u>Proof</u>: We set $g(y) := f^{-1}(y)$

Then $$\frac{g(y+h) - g(y)}{h} = \frac{x+k-x}{f(x+k) - f(x)} = \frac{k}{f(x+k) - f(x)}$$

$$= (1) \Big/ \left(\frac{f(x+k) - f(x)}{k}\right) \xrightarrow{k \to 0} = \frac{1}{f'(x)} \leftarrow x = g(y)$$

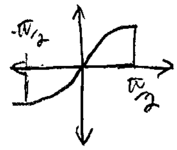


$$[f^{-1}(y)]' = \frac{1}{f'(f^{-1}(y))}$$

$$= \frac{1}{f(g(y))}$$

$$\left[= \frac{1}{f'(f^{-1}(y))} \right]$$

Example. Consider $\sin : (-\frac{\pi}{2}, \frac{\pi}{2}) \to (-1, 1)$



one has $\sin'(x) = \cos(x) > 0$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

↳ $\sin$ is strictly increasing on $(-\frac{\pi}{2}, \frac{\pi}{2})$

We call its inverse arcsin.

One has $\arcsin(y)' = \frac{1}{\cos(\arcsin(y))}$

Since $\cos^2(\cdot) = 1 - \sin^2(\cdot)$ one has

$$= \frac{1}{\sqrt{1 - \sin^2(\arcsin(y))}}$$

$$= \frac{1}{\sqrt{1 - y^2}} \quad \text{for } y \in (-1, 1).$$

VII Integration AIM: find an operation which is the inverse of taking the derivative. Compute some integrals.

VII.1 The Indefinite Integral

Def: Let $f : I \to \mathbb{R}$ be a function. An indefinite integral (aka antiderivative) is a differentiable function $F : I \to \mathbb{R}$ such that $F' = f$

(don't use it?)

Example

1) If $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^n$, $n \in \mathbb{N}$, then $F(x) = \frac{1}{n+1} x^{n+1} + C$ for any $C \in \mathbb{R}$ is an indefinite integral for f

2) $f : \mathbb{R}_+ \to \mathbb{R}$, $f(x) = x^{\alpha}$ with $\alpha \in \mathbb{R} \setminus \{-1\}$.
Then $F(x) = \frac{1}{\alpha+1} x^{\alpha+1} + C$ is an ind. int for f for any $C \in \mathbb{R}$

3) If $f : \mathbb{R}_+ \to \mathbb{R}$, $f(x) = \frac{1}{x}$ then $F(x) = \ln(x) + C$

Remarks

1) If an indefinite integral for f exists, and we denote it by F, then $F + c$ is also an indefinite integral for f

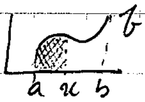
2) If F_1 and F_2 are indefinite integrals for f. Then $(F_1 - F_2)' = f - f = 0$

Since only a constant function has a derivative equal to 0 on an interval, one gets $F_1 - F_2 = cte \iff F_1 = F_2 + cte$

3) We write $\int f(x)\,dx$ or $\int f$ for the indefinite integral

VII.2 Some Areas

Consider $f : [a,b] \to \mathbb{R}_+$ be continuous.

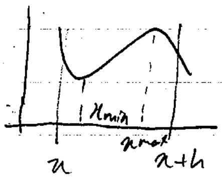


For any $x \in [a,b]$, we set $F(x) :=$ surface between x-axis and curve between a and x

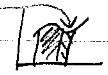
In particular $F(a) = 0$

Theorem The function $F : [a,b] \to \mathbb{R}_+$ is differentiable and its derivative is equal to f. namely $F'(x) = f(x)$ $\forall x \in (a,b)$

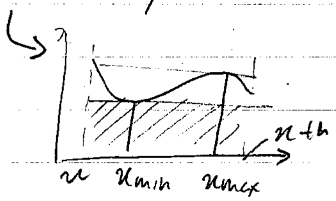
Remark: Observe that F is not really well-defined since the notion of surface has not been defined!



Proof let us fix $x \in (a,b)$ and let $h > 0$ such that $x + h \in (a,b)$, then: $\dfrac{F(x+h) - F(x)}{h}$



☆ Since f is continuous on $[x, x+h]$ then there exist $x_{min}, x_{max} \in [x, x+h]$



Thus $h(f(x_{min})) \leq F(x+h) - F(x) \leq h(f(x_{max}))$

$$\Leftrightarrow \quad f(x_{min}) \leq \frac{F(x+h) - F(x)}{h} \leq f(x_{max})$$

taking the limit $h \to 0$, one gets

$$\left.\begin{array}{l} \lim_{h\to 0} f(x_{min}) = f(x) \\ \lim_{h \to 0} f(x_{max}) = f(x) \end{array}\right\} \text{ and thus: } \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = f(x)$$

$\square$

☆ **Consequence**

The function $F : (a,b) \to \mathbb{R}_+$ is an indefinite integral for f.

The area below the curve between a and b is given by $F(b)$

☆ **Corollary**

If G is an indefinite integral for the same function f, then:

$$G(b) - G(a) = F(b) - \underbrace{F(a)}_{=0}$$

$=$ the surface below the curve between a and b

Proof Since $G = F + C$ for some $C \in \mathbb{R}$ then $G(b) - G(a) = (F(b) + C) - (F(a) + C)$

$$= F(b) - \underbrace{F(a)}_{=0} = F(b).$$

No.

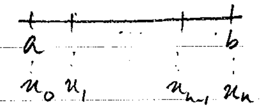
Date

VII.3 Rieman's Sum.

Question: How can one construct an indefinite integral for "general" function?

Def: For any interval $[a,b]$ we call a <u>n-partition P of $[a,b]$</u>, a set of points $\{x_j\}_{j=0}^{n}$ with $a = x_0 < x_1 < x_2 < \dots < x_n = b$

Example

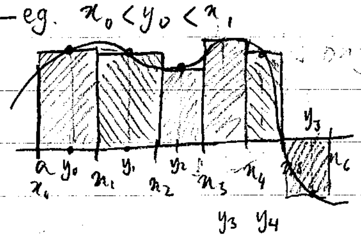
i) 

number of partition

Def: For any $f: [a,b] \to \mathbb{R}$, bounded, for any n-partition P and for any family $\{y\}_{j=0}^{n-1}$ with $x_j \leq y_j < x_{j+1}$, we set:

$$\underbrace{R(\{y\}_{j=0}^{n-1}, P, f) = \sum_{j=0}^{n-1} f(y_j)(x_{j+1} - x_j)}_{\text{Rieman's Sum.}}$$

eg. $x_0 < y_0 < x_1$



☆ Idea: we shall consider finer partitions (=smaller subintervals).

▷ Remark: We can choose the points y_j such that $f(y_j)$ is <u>maximal</u> ① in the interval $[x_j, x_{j+1}]$, or such that $f(y_j)$ is <u>minimal</u> in the interval $[x_j, x_{j+1}]$. ②

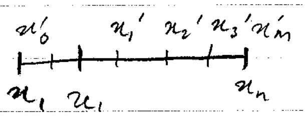
▶ The first choice is called the <u>upper sum</u>, denoted by $R_{(max)}(f, P)$

▶ The secon choice is called the lower sum, —— " —— $R_{(min)}(f, P)$

▷ by <u>increasing</u> the partitions of upper sum, it decreases

▷ by <u>increasing</u> the partitions of lower sum, it increases.

> Consider a partition P' which finer than P.



Then $R_{(max)}(f, P') \leq R_{(max)}(f, P)$
$\qquad\; \vee\!\shortparallel \qquad\qquad\quad \vee\!\shortparallel$
$\quad R_{(min)}(f, P') \geq R_{(min)}(f, P)$

→ If M is the maximum value of f on $[a,b]$ and m the minimum val on of f on $[a,b]$, Then:

$$m(b-a) \leq R_{min}(f,P) \leq R_{\{y_j\}}(f, P) \leq R_{(max)}(f,P) \leq M(b-a)$$

Def A bounded function $f: [a,b] \to \mathbb{R}$ is a Riemann Integrable

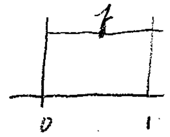
if $\underbrace{\sup}_{\text{supremum}} R_{(min)}(f,P) = \underbrace{\inf}_{\text{infimum}} R_{(max)}(f,P)$ → we write $\int_a^b f(x)dx$ for this number obtained.

increasing P partitions, $R\uparrow$ ⟵ converge? ⟶ $R\downarrow$ increasing P partitions

Example (of a non integrable function)

$f: [0,1] \to [0,1]$

$$f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \in \mathbb{R}/\mathbb{Q} \end{cases}$$



Let P be any n-partition of $[0,1]$

$R_{min}(f,P) = 0$

$R_{max}(f,P) = 1$

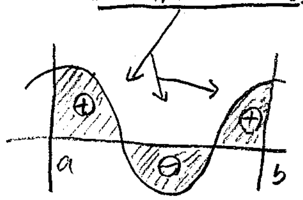
these two will never meet even if P becomes finer and finer. So it's non Riemann inte.

Theorem If $f: [a,b] \to \mathbb{R}$ is continuous, then f is Riemann Integrable.

($\Rightarrow \int_a^b f(x)\,dx$ is well defined)

Remark → If $f: [a,b] \to \mathbb{R}_+$ is continuous, the construction corresponds to what has been done in Section VII.2.

↳ but if f changes its sign, then the expression $\int_a^b f(x)dx$ computes the oriented area.



Properties Assume that f is Riemann integrable

1) If a —— c —— b, then:

then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

2) If $a < b$, then we set:

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Then for any $\underbrace{a, b, c \in \mathbb{R}}_{\text{the order of } a, b, c \text{ is arbitrary.}}$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

this space is intentionally left blank.

Theorem: (extension of the prev. lesson's proof).

For $f: [a,b] \to \mathbb{R}$, continuous, let us set for any $x \in [a,b]$:

$$F(x) = \int_a^x f(y)\,dy \quad \left(\begin{array}{l} f \text{ is Riemann integrable} \\ \text{on } [a,x] \text{ since it's continuous} \end{array} \right)$$

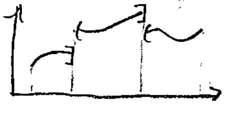
Then $F: [a,b] \to \mathbb{R}$ is differentiable on (a,b) and $F'(x) = f(x)$

This is called the Fundamental Theorem of Calculus.

Proof: similar to the previous one (last week). □

Remark: This statement is also correct for more general function (not continuous) but the proof is more complicated)

If f is continuous except at a finite number of points, then the statement and the proof are (almost) similar



VII.4 Properties of the Riemann Integral.

Lemma: 1) if $f, g : [a,b] \to \mathbb{R}$ are Riemann integrable, then $\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_c^b g(x)\,dx$

2) $\int (\lambda f)(x)\,dx = \lambda \int_a^b f(x)\,dx$

3) $\int_a^b f(x) g(x)\,dx \neq \int_a^b f(x)\,dx * \int_a^b g(x)\,dx$

4) if $f(x) \geq g(x)$ $\forall x \in [a,b]$ then $\int_a^b f(x)\,dx \geq \int_c^b g(x)\,dx$

5) if $f : [a,b] \to \mathbb{R}$ is Riemann integrable then $|f|$ is also integrable, and $\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$

easy proof if f is continuous $\Rightarrow$ if $f(x)$ is continuous, then $|f(x)|$ is also continuous

VII.5 Improper Integral → integration where the bounds may be not definite.

Consider the integrals for $a \in (0,1)$

1) $= \int_a^1 \frac{1}{y}\,dy$

$= \ln(1) - \ln(a)$

$= -\ln(a)$

$\Rightarrow \lim_{a \to 0} -\ln(a) =$ does not exist.

2) $= \int_a^1 \frac{1}{y^{1/2}}\,dy$

$= 2y^{1/2} \big|_a^1$

$= 2(1 - \sqrt{a})$

$\Rightarrow \lim_{a \to 0} 2(1-\sqrt{a}) = 2$

Similarly consider for $b > 1$

$\int_1^b \frac{1}{y}\,dy$

$= \ln(b)$

$\Rightarrow \lim_{b \to \infty} \ln(b) =$ does not exist

$\int_1^b \frac{1}{y^2}\,dy$

$= -y^{-1} \big|_1^b$

$= 1 - \frac{1}{b}$

$\Rightarrow \lim_{b \to \infty} 1 - \frac{1}{b} = 1$

☆ Def: Let $f : (a,b] \to \mathbb{R}$ be continuous. If $\lim_{x \to a+} \int_x^b f(y)\,dy$ exists, we write $\int_a^b f(y)\,dx$ for $\lim_{x \to a+} \int_x^b f(y)\,dy$.

Similarly, if $g : [a, \infty) \to \mathbb{R}$ is continuous and if $\lim_{b \to \infty} \int_a^b g(y)\,dy$ exists, then we write $\int_a^\infty g(y)\,dy$.

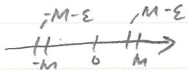
→ These integrals are called improper Riemann integrals. The same holds for $\lim_{x \to a-}$ or $\lim_{x \to \infty}$.

?) Question: what is $\int_{-\infty}^{\infty} f(y)\,dy$? This can be defined by $\lim_{b \to +\infty} \lim_{a \to -\infty} \int_a^b f(y)\,dy$ if it exists, or $\int_{-\infty}^{c} f(y)\,dy + \int_c^\infty f(y)\,dy$ if they exist.

⚠ $\Rightarrow \int_{-\infty}^{\infty} f(y)\,dy$ can not be defined by $\lim_{M \to \infty} \int_{-M}^{M} f(y)\,dy$

↳ for example $\int_{-M}^{M} \sin x\,dx = 0 \Rightarrow \lim_{M \to \infty} \int_{-M}^{M} \sin(y)\,dy = 0$

☆ Consider for $\varepsilon > 0$



$\int_{-M-\varepsilon}^{M-\varepsilon} \sin(y)\,dy = \cos(M+\varepsilon) - \cos(M-\varepsilon)$

⚠ for the function $\sin$ $\lim_{b \to \infty} \int_a^b \sin(y)\,dy$ has no limit

end of chapter

Techniques of Integration

VIII Techniques of Integration

VIII. Substitution assume that $\int f(x)\,dx = \int g'(u(x))\,u'(x)\,dx = G(u(x))$

if G is an indefinite integral for g.

$\star = \int_a^b g(u(x))\,u'(x)\,dx$
$= \int_a^b g(u(x))\,\frac{du}{dx}(x)\,dx$

Examples

$\rightarrow \int (x^3+x)^9(3x^2+1)\,dx = \frac{1}{10}(x^3+x)^{10} + C$

$\rightarrow \int x\sin(x^2)\,dx = \frac{1}{2}\int 2x\sin(x^2)\,dx$
$= -\frac{1}{2}\cos(x^2) + C$

$\star$ When we do a substitution, we look for a function u and set $du = \frac{du}{dx}(x)\,dx$

$\star \int_a^b g(u(x))\,u'(x)\,dx = \int_a^b g(u(x))\,\underline{\frac{du}{dx}(x)\,dx}$

$\| \quad G(u(b)) - G(u(a)) = \int_{u(a)}^{u(b)} g(u)\,\underline{du}$

Example

$\rightarrow \int x^5\sqrt{1-x^2}\,dx = \int (1-u)^{5/2}\sqrt{u}\,dx$

set $u = 1-x^2$
$du = -2x\,dx$
$x^2 = 1-u$
$du = -2\sqrt{1-u}\,dx$
$dx = \frac{du}{-2\sqrt{1-u}}$

$= \int (1-u)^{5/2}\sqrt{u}\,\frac{du}{-2\sqrt{1-u}}$
$= -\frac{1}{2}\int (1-u)^2 u^{1/2}\,du$
$= -\frac{1}{2}\int (1-2u+u^2)u^{1/2}\,du$
$= -\frac{1}{2}\int (u^{1/2} - 2u^{3/2} + u^{5/2})\,du$
$= -\frac{1}{2}\left[\frac{2}{3}u^{3/2} - 2\,\frac{2}{5}u^{5/2} + \frac{2}{7}u^{7/2}\right] + C$
$= -\frac{1}{3}(1-x^2)^{3/2} + -\frac{2}{5}(1-x^2)^{5/2} + \frac{2}{7}(1-x^2)^{7/2} + C$

VIII.2 Integration by Parts.

Recall that $(fg)' = f'g + g'f$
$\Rightarrow \int (fg)' = \int (f'g + g'f)$
$\int (fg)'(x)\,dx = \int f'g + \int g'f$

$\Longleftrightarrow \boxed{\int f'g = fg - \int g'f}$

Examples

$\rightarrow \int xe^x\,dx = xe^x - \int e^x\,dx$
let $f'(x) = e^x$, $g(x) = x$
$= xe^x - e^x + C$

$\rightarrow \int \ln(x)\,dx = \int 1\,\ln(x)\,dx$
let $g(x) = \ln x$, $f'(x) = 1$
$= x\ln x - \int x\frac{1}{x}\,dx$
$= x\ln x - x + C$

VIII Trigonometric integrals.

Remember that:
- $\sin^2(x) + \cos^2(x) = 1$
- $\sin^2(x) = \frac{1-\cos(2x)}{2}$
- $\cos^2(x) = \frac{1+\cos(2x)}{2}$

Example:

$\rightarrow \int \sin^2(x)\,dx = \int \frac{1-\cos(2x)}{2}\,dx$
$= \frac{1}{2}\int 1-\cos 2x\,dx$
$= \frac{1}{2}x - (\sin 2x)\frac{1}{4} + C$

No.

Date

example - continued.

$= \int \cos^3(x)\,dx =$

$= \int \cos^2(x) \cos(x)\,dx$

$= \int (1-(\sin(x))^2) \cos(x)\,dx$

$= \sin(x) - \frac{1}{3}(\sin(x))^3 + C$

VIII.4 Partial Fraction

Examples

$\rightarrow \int \frac{1}{x-\alpha}\,dx = \ln(x-\alpha) + C$

$\rightarrow \int \frac{1}{(x-\alpha)^n} = \int (x-\alpha)^{-n}\,dx = \frac{1}{-n+1}(x-\alpha)^{-n+1} + C$

$n \neq 1$

$\rightarrow \int \frac{1}{x^2+x}\,dx = \arctan(x) + C$,

☆ More generally, if we consider $\frac{f(x)}{g(x)}$ with f, g polynomials with degree$(f) \leq$ degree(g). Then one has to factor the denominator into terms of the form $(x-\alpha_j)^n$ and $((x-\beta_j)^2+\gamma_j)^m$

☆ More precisely, one has to write $\frac{f(x)}{g(x)} = \frac{C_1}{x-\alpha_1} + \frac{C_2}{(x-\alpha_2)^2} + \cdots + \frac{C_n}{(x-\alpha_n)^n} + \cdots$

$+ \frac{a_1+b_1x}{(x-\beta_1)^2+\gamma_1} + \cdots + \frac{a_m+b_mx}{((x-\beta_m)^2+\gamma_m)^m}$

→ example of a trick.

$\int \frac{1}{(x^2+1)^2}\,dx = ?$

Solution

$\hookrightarrow \underbrace{\int 1 \cdot \frac{1}{x^2+1}\,dx}_{=\arctan(x)}$

$= \frac{x}{x^2+1} + 2\int \frac{x \cdot x}{(x^2+1)^2} + \frac{1-1}{(x^2+1)^2}\,dx$

$= \frac{x}{x^2+1} + 2\int \frac{x^2+1-1}{(x^2+1)^2}\,dx$

$= \frac{x}{x^2+1} + 2\int \frac{x^2+1}{(x^2+1)^2}\,dx + 2\int \frac{-1}{(x^2+1)^2}\,dx$

$= \frac{x}{x^2+1} + \underbrace{2\int \frac{x^2+1}{(x^2+1)^2}\,dx}_{=2\int \frac{1}{(x^2+1)}\,dx = 2\arctan x} - 2\int \frac{1}{(x^2+1)^2}\,dx$ this is what we're looking for.

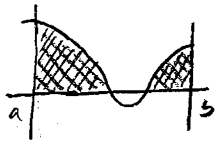
$\Longrightarrow \int \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\arctan(x) + \text{constant}$

Reminder

Let $f:[a,b]\to\mathbb{R}$ be continuous, and $x\in[a,b]$. The limit of Riemann sum on $[a,x]$ is an indefinite integral for f on $[a,x]$ and if we set $F(x)$ for its limit, then

$$F(x) = F(x) - \underbrace{F(a)}_{=a} = \int_a^x f(y)\,dy$$
$$= G(x) - G(a) = G(y)\Big|_a^x$$

if G is any other indefinite integral.



IX Taylor Expansion

Idea: approximate a function by a polynomial, at least locally.

Recall: that if $f: I\to\mathbb{R}$ sufficiently differentiable many times.

we write: $f^{(0)} = f$
$f^{(1)} = f'$
$f^{(2)} = (f')' = f''$
$\vdots$
$f^{(n)}$

Definition Consider $f:[a,b]\to\mathbb{R}$, sufficiently many times differentiable, and let $x_0\in[a,b]$. For any $x\in[a,b]$ we write:

$$P_n(x_0,x) = f(x_0) + \frac{1}{1!}f'(x_0)(x-x_0)^1 + \frac{1}{2!}f^{(2)}(x_0)(x-x_0)^2$$
$$+ \dots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n$$
$$= \sum_{j=0}^{n}\frac{1}{j!}f^{(j)}(x_0)(x-x_0)^j$$

and we call this Taylor polynomial of f at x with respect to x_0 and of degree n

Remark

$P_n(x_0,x_0) = f(x_0)$
$P_n'(x_0,x_0) = f'(x_0)$
$P_n''(x_0,x_0) = f''(x_0)$
$\vdots$
$P_n^{(n)}(x_0,x_0) = f^{(n)}(x_0)$

It means the n^{th} first derivatives of $P_n(x_0,\bullet)$ and f evaluated at x_0 are the same.

Example

1) $f(x) = \sin(x)$ and $x_0 = 0$, $n=5$

$$P_5(0,x) = \sin(0) + \frac{1}{1}\cos(0)(x) - \frac{1}{2}\sin(0)(x)^2 - \frac{1}{3!}\cos(0)x^3 + \frac{1}{4!}\sin(0)x^4$$
$$+ \frac{1}{5!}\cos(0)x^5$$

$$P_5(0,x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

2) $f(x) = e^x$, $x_0 = 0$, n arbitrary

$$P_n(0, x) = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots + \frac{1}{n!}x^n$$

What $\sin(x) - P_8(0,x)$ or $e^x - P_n(0, x)$?

Theorem → Taylor Expansion Theorem

▲ Let $f : [a,b] \to \mathbb{R}$ b $(n+1)$ times differentiable, with $f^{(n+1)}$ continuous on $[a,b]$

↳ notation: $\Leftrightarrow$ $f \in C^{n+1}([a,b])$

▲ let $x_0 \in [a,b]$. Then for any $x \in [a,b]$:

$$\star \quad f(x) = P_n(x_0, x) + R_{n+1}(x_0, x)$$

with $R_{n+1}(x_0, x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt$

$$= \frac{1}{(n+1)!} f^{(n+1)}(c)(x - x_0)^{n+1} \quad \text{some } c \in (x_0, x)$$

↳ Proof (proof by Induction).

If $n = 0$, then the statement is

$$f(x) = \underset{P_0(x_0,x)}{\underset{\shortparallel}{f(x_0)}} + \int_{x_0}^{x} f'(t)\, dt \quad ✓ \qquad \int_{x_0}^{x} \frac{(x-t)^0}{0!} f^{(0+1)}(t)\, dt = \int_{x_0}^{x} \frac{1}{1} f'(t) dt$$

Thus the statement is correct for $n=0$. So, let us assume by induction that it is true for $n-1$, and let us deduce it for n, with $n \geq 1$.

So we have $f(x) = P_{n-1}(x_0, x) + R_n(x_0, x)$, with: $R_n(x_0, x) = \int_{x_0}^{x}$

$$R_n(x_0, x) = \int_{x_0}^{x} \underbrace{\frac{(x-t)^{(n-1)}}{(n-1)!}}_{v'(t)} \underbrace{f^{(n)}}_{u(t)}(t)\, dt$$

$\Downarrow$

$$v(t) = -\frac{(x-t)^n}{n!}$$

$$= -\frac{(x-t)^n}{n!} f^{(n)}(t)\, dt + \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, dt$$

$$+ \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) + R_{n+1}(x_0, x)$$

One has obtained

$$f(x) = \underbrace{P_{n-1}(x_0, x) + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0)} + R_{n+1}(x_0, x)$$

The statement is true for n arbitrary.

continued →

we have closed interval
cuz we need a max value
without a closed interval.
Open interval has no max.

☆ For the second expression, recall that $f^{(n+1)}$ is continuous on $\underbrace{[a,b]}$

$\Rightarrow \exists\, x_{min}$ and x_{max} such that $f^{(n+1)}(x_{min}) \leq f^{(n+1)}(y) \leq f^{(n+1)}(x_{max})$

for any $y \in [a,b]$

Thus, for any $t \in [x_0, x]$,

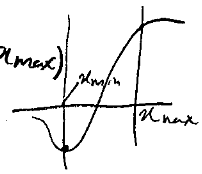
$$\frac{(x-t)^n}{n!} f^{(n+1)}(x_{min}) \leq \frac{(x-t)^n}{n!} f^{(n+1)}(t) \leq \frac{(x-t)^n}{n!} f^{(n+1)}(x_{max})$$

Let's integrate with respect to t

$$\Rightarrow f^{(n+1)}(x_{min}) \int_{x_0}^{x} \frac{(x-t)^n}{n!}\,dt \leq \underbrace{\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt}_{R_{n+1}(x_0,x)} \leq f^{(n+1)}(x_{max}) \int_{x_0}^{x} \frac{(x-t)^n}{n!}\,dt$$

$$\Rightarrow f^{(n+1)}(x_{min}) \int_{x_0}^{x} \frac{(x-t)^n}{n!}\,dt \leq R_{n+1}(x_0,x) \leq f^{(n+1)}(x_{max}) \int_{x_0}^{x} \frac{(x-t)^n}{n!}\,dt$$

Since $f^{(n+1)}$ takes all values between $f^{(n+1)}(x_{min})$ and $f^{(n+1)}(x_{max})$, there exist $c \in [x_0, x]$ such that:



$$f^{(n+1)}(c) \underbrace{\int_{x_0}^{x} \frac{(x-t)^n}{n!}\,dt} = R_{n+1}(x_0,x)$$

$$= -\left.\frac{(x-t)^{n+1}}{(n+1)!}\right|_{x_0}^{x} = \frac{(x-x_0)^{n+1}}{(n+1)!}$$

$$\Leftrightarrow f^{(n+1)}(c)\, \frac{(x-x_0)^{n+1}}{(n+1)!} = R_{n+1}(x_0,x).$$ ⌧ end of proof.

__Remark__

$$|R_{n+1}(x_0,x)| = |f^{(n+1)}(c)| \frac{|(x-x_0)|^{n+1}}{(n+1)!} \quad \text{for some } c \in [x_0, x]$$

$$\leq \frac{|x-x_0|^{n+1}}{(n+1)!} \cdot \max\left\{ |f^{(n+1)}(x_{min})|, |f^{(n+1)}(x_{max})| \right\}$$

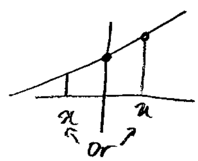
__Example__

1) $|\sin(x) - P_n(0,x)| = |R_{n+1}(0,x)| \leq \frac{|x|^{n+1}}{(n+1)!}(1)$ since $|\sin^{(n+1)}(y)| \leq 1$

2) $|e^x - P_n(0,x)| = |R_{n+1}(0,x)| \leq \frac{|x|^{n+1}}{(n+1)!} \begin{cases} e^x & \text{if } x > 0 \\ 1 & \text{if } x < 0 \end{cases}$

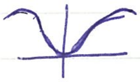
__Remark__

For any $x \in \mathbb{R}$, one has $\frac{|x-x_0|^{n+1}}{(n+1)!} \xrightarrow{n \to \infty} 0$

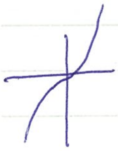


Remark

If $x=0$, consider an even function $\Leftrightarrow f(-x) = -f(x)$



all coefficient of the polynomial with odd powered terms will disappear,



the opposite happens with ~~even~~ odd function

<u>Reminder</u> $f:[a,b]\to\mathbb{R}$ sufficiently differentiable. Let $x_0\in(a,b)$ and set for $n\in\mathbb{N}$, $P_n(x_0,x):=\sum_{j=0}^{n}\frac{1}{j!}f^{(j)}(x_0)(x-x_0)^j$

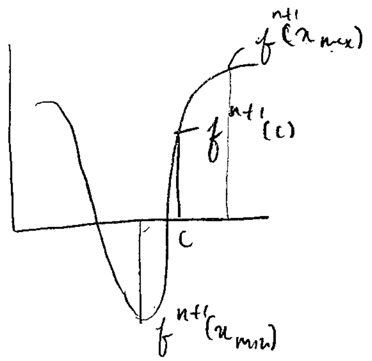
Then $f(x)=P_n(x_0,x)+R_{n+1}(x_0,x)$ with:

with $R_{n+1}(x_0,x)=\int_{x_0}^{x}\frac{(x-t)^n}{n!}f^{(n+1)}(t)\,dt$

$=\frac{1}{(1+n)!}f^{(n+1)}(c)\,(x-x_0)^{n+1}$ for some $c\in(x_0,x)$

Then $|R_{n+1}(x_0,x)|\leq\frac{|x-x_0|^{n+1}}{(n+1)!}\max\{|f^{n+1}(x_{max})|,|f^{n+1}(x_{min})|\}$

for $x_{min}, x_{max}\in(x_0,x)$.



$f^{n+1}(c)\leq|f^{n+1}(x_{max})|$ or $|f^{n+1}(x_{min})|$

Taylor expansion at $x_0=0$ for <u>odd</u> or <u>even function</u>.

$f(-x)=-f(x)\ \forall x\in\mathbb{R}$ $\qquad$ $f(-x)=f(x)\ \forall x\in\mathbb{R}$

Recall that if f is differentiable

$$\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=f'(x)$$

→ What about the derivative of an even/odd function?

<u>Lemma</u>: If f is an odd function, then f' is even and if f is even, then f' is odd.

<u>Proof:</u>

1) Let f be odd.

$$\frac{f(x+h)-f(x)}{h}\xrightarrow{h\to0}f'(x)$$

$$=\frac{-f(-(x+h))--f(-x)}{h}=\frac{-[f(-x-h)-f(-x)]}{h}$$

$$=\frac{f(-x-h)-f(-x)}{-h}\xrightarrow{h\to0}f'(-x)$$

$\therefore f'(x)=f'(-x)$, hence f' is even.

2) let f be even

$$\frac{f(x+h)-f(x)}{h} \xrightarrow{h\to 0} f'(x)$$

$$= \frac{f(-(x+h)) - f(-x)}{h} = \frac{f(-x-h) - f(-x)}{h}$$

$$= - \frac{f(-x-h) - f(-x)}{-h} \xrightarrow{h\to 0} -f'(-x)$$

$f'(x) = -f'(-x)$ it's an odd function

<u>Remark</u> if f is odd, then $f(0)=0$. Indeed, $f(x) = -f(-x)$ $\forall x \in \mathbb{R}$ in particular for $x=0$,

$$f(0) = -f(0) \Rightarrow f(0) = 0$$

<u>Theorem</u> let f be suff. differentiable and consider $P_n(0,x)$ centered at 0 ($\Uparrow$ Taylor's Poly.)

$$P_n(0,x) = \sum_{j=0}^{n} \frac{1}{j!} f^{(j)}(0)\, x^j$$

→ if f is even, then $f^{(j)}(0) = 0$ for <u>odd j</u>

→ if f is odd, then $f^{(j)}(0) = 0$ for <u>even j</u>.

<u>Proof</u>

→ if f is even, then $f^{(1)}, f^{(3)}, f^{(5)}, \dots$ are odd functions

$\Rightarrow$ ~~for any~~ $f^{(j)}(0) = 0$ for any <u>odd j</u>

→ if f is odd, then $f, f^{(2)}, f^{(4)}, f^{(6)}, \dots$ are even functions

$\Rightarrow f^{(j)}(0) = 0$ for any <u>even j</u>

end of proof

Remember that this only applicable for Taylor's Expansion centered around "0"! Remember!!

2018

X Series ($\neq$ sequence)

Natural Question: can we write $f(x) = \sum_{j=0}^{\infty} \frac{1}{j!} f^{(j)}(x_0)(x-x_0)^j$?

X.1 Convergent series

↳ Let $a_1, a_2, a_3, \dots$ be real numbers, we write $(a_j)_{j\in\mathbb{N}}$ for such a sequence of numbers.

↳ A series is a sum of the form $\sum_{j=1}^{\infty} a_j$. Rewrite: $\sum_{j=1}^{n} a_j$ ← ✓

Definition The series $a_1 + a_2 + a_3 + \dots$ converges if the partial sum $S_n := \sum_{j=1}^{n} a_j$ converges when n goes to infinity, it means if $\lim_{n\to\infty} S_n$ exists. おわり。

Definition of [$\lim_{n\to\infty} S_n$]: It means there exist $S_\infty \in \mathbb{R}$ such that $\forall \varepsilon > 0$, $\exists N > 0$ with $|S_n - S_\infty| \leq \varepsilon$ for any $n \geq N$

If S_n converge when n goes to infinity, we write $\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} S_n = S_\infty$. We also say that the series is convergent.

Stupid Example

1) Let $a_j = \begin{cases} 1 & \text{if } j \text{ is even} \\ -1 & \text{if } j \text{ is odd} \end{cases}$ Then, is $\sum_{j=1}^{\infty} a_j$ convergent?

Let's compute
$S_1 = -1$
$S_2 = 0$
$S_3 = -1$
$S_4 = 0$
$\vdots$

The sequence of S_n is not convergent
$\Rightarrow$ the series of S_n is not convergent

Better example

2) Consider $a_j = \frac{1}{2^j}$ $\Rightarrow$ $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{4}$, $a_3 = \frac{1}{8}$, $\vdots$ One has $S_n = \sum_{j=1}^{n} a_j$

$$S_n = \sum_{j=1}^{n} \frac{1}{2^j} \quad \text{let } k := j-1$$
$$= \sum_{k=0}^{n-1} \frac{1}{2^{k+1}}$$
$$= \sum_{k=0}^{n-1} \frac{1}{2} \cdot \frac{1}{2^k}$$
$$= \frac{1}{2} \sum_{k=0}^{n-1} \frac{1}{2^k} = \frac{1}{2} \cdot \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}$$
$$= \cancel{\frac{1}{2}} \cdot \frac{1-(\frac{1}{2})^n}{\cancel{\frac{1}{2}}}$$
$$= 1 - \left(\frac{1}{2}\right)^n$$

take $n\to\infty$

since $S_n \xrightarrow{n\to\infty} 1$, it means that the series is convergent and converges to 1

3) $a_j = \frac{1}{j}$ What about $\sum_{j=1}^{\infty} \frac{1}{j} = 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{2}} + \underbrace{\frac{1}{9} + \dots + \frac{1}{16}}_{\geq \frac{1}{2}}$

$\sum_{j=1}^{\infty} \frac{1}{j}$

$\geq 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} \dots$

$\hookrightarrow \to \infty$

if you pair enough times, you'll get more & more numbers greater than $\frac{1}{2}$, hence it does not converge. $\square$

⚠ We denote by $(a_j)_{j \in \mathbb{N}}$ as sequence, and $\sum_{j=1}^{\infty} a_j$ is called the associated series.

Lemma: Let $(a_j)_{n \in \mathbb{N}}$, $(b_j)_{j \in \mathbb{N}}$ with converging associated series.

1) $\sum_{j=1}^{\infty} \lambda a_j = \lambda \sum_{j=1}^{\infty} a_j$ is convergent.

2) $\sum_{j=1}^{\infty} (a_j + b_j) = \sum_{j=1}^{\infty} a_j + \sum_{j=1}^{\infty} b_j$ & you can separate them when they have finite sum first.

3) $\sum_{j=1}^{\infty} a_j b_j = \quad \neq \left(\sum_{j=1}^{\infty} a_j\right)\left(\sum_{j=1}^{\infty} b_j\right)$

$\hookrightarrow$ ~~Sum~~ products of the 2 convergent series of does not always convergent.

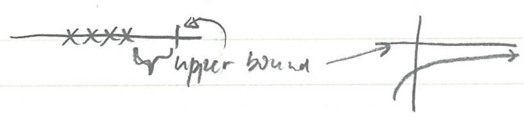
X.2 Series with positive terms only.

→ It means each $a_j \geq 0$. This implies $S_{n+1} \geq S_n$ and thus the partial sums are increasing.

→ Then we can use:

Theorem: (monotone convergence theorem)

$\hookrightarrow$ Any increasing sequence $(S_n)_{n \in \mathbb{N}}$ which is upper bounded is convergent. $\exists C$ such $S_n \leq C$ for any n



Example Consider $a_j = \frac{1}{j^2}$

$$\sum_{j=1}^{\infty} a_j = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots = 1 + \underbrace{\frac{1}{2^2} + \frac{1}{3^2}} + \underbrace{\frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2}} + \frac{1}{8^2} + \dots$$

$$\leq 1 + \frac{1}{2^2} + \frac{1}{2^2} + \frac{1}{4^2} \times 4 + 8 \frac{1}{8^2} + \dots$$

$$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1 + \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^j = 2$$

In fact, $\underbrace{\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}}_{\text{Basel problem}}$

☆ $\sum_{j=1}^{\infty} \frac{1}{j^n} = ?$

"Life is simple if we only have positive number" – Serge Richard.

comparison

① **Corollary** Let $(a_j)_{j\in\mathbb{N}}$, $(b_j)_{j\in\mathbb{N}}$ be 2 sequences of positive numbers, and assume that $\sum_{j=1}^{\infty} b_j < \infty$. Then if $\exists\, d>0$ such that $a_j \leq d\cdot b_j$, then $\sum_{j=1}^{\infty} a_j \leq d \sum_{j=1}^{\infty} b_j$

② **Corollary**: If $a_j \geq d\cdot b_j$ and if $\sum_{j=1}^{\infty} b_j$ is not convergent, then $\sum_{j=1}^{\infty} a_j$ is not convergent either.

Remark 2 additional criteria for convergence will be seen in Homework 14.

X.3 Absolute convergence and series with positive and negative terms

⚠ When terms are positive and negative, we have to be careful:

for example $\sum_{j=1}^{\infty} (-1)^j \frac{1}{j}$ or $1-1+1-1+1$ which converges to $\frac{1}{2}$

$\Downarrow$ is convergent

OR

OR $\sum_{j=1}^{\infty} (-1)^j j! = 1-1+2-6+24-120\cdots$
$\cong 0.596374$

definition of 「Absolute Convergence」

↳ A series $a_1+a_2+a_3+\ldots$ is absolutely convergent if $\sum_{j=1}^{\infty} |a_j| < \infty$

☆ absolute convergent defines a series as convergent, but convergent does not always define absolute convergent.