

Epilogue

Recall $M(\mathbb{R}^d)$ (all complex Radon measures on \mathbb{R}^d) = $C^c(\mathbb{R}^d)^* \ni \varphi$

$$\left(\int \varphi(x) \mu(dx)\right) \in \mathbb{C}$$

For $\mu \in M(\mathbb{R}^d)$, $f_1, \dots, f_n \in C^c(\mathbb{R}^d)$, $\varepsilon > 0$, we set

$$U(\mu, f_1, \dots, f_n, \varepsilon) = \{v \in M(\mathbb{R}^d) \mid |\mu(f_j) - v(f_j)| < \varepsilon \forall j=1, \dots, n\}$$

$\{U(\mu, f_1, \dots, f_n, \varepsilon) \mid f_1, \dots, f_n \in C^c(\mathbb{R}^d), \varepsilon > 0\}$ is a ngd. base for μ in $M(\mathbb{R}^d)$, and we call the corresponding topology the weak* topology.

(or the vague topology)

Recall that for $\mu \in M(\mathbb{R}^d)$ for $R > 0$ we set

$$\gamma_\mu^R = \frac{1}{\text{vol}(B_R)} \mu|_{B_R} * \tilde{\mu}|_{B_R} \text{ (in [GB]) or } \gamma_\mu^R = \frac{1}{L^d} \mu|_{C_L} * \tilde{\mu}|_{C_L} \text{ (in Hof)}$$

Lemma

If $\mu \in M(\mathbb{R}^d)$ is translation bounded, then

then γ_μ^R is translation bounded

and $\{\gamma_\mu^R\}_{R>0}$ is precompact in $M(\mathbb{R}^d)$ for the weak* topology

$\hookrightarrow K := \overline{\{\gamma_\mu^R\}_{R>0}} \text{ (closure in } w^* \text{-topology) is compact}$

Corollary

Since $\{\gamma_\mu^R\}$ is a net in K , which is compact,

it has at least one convergent subnet.

We call the limit of this convergent subnet an autocorrelation for μ .

"Every accumulation point of $\{\gamma_\mu^R\}_{R>0}$ in vague topology is called an autocorrelation for μ . If $\lim_{R \rightarrow \infty} \gamma_\mu^R$ exists in the vague topology, we call it the natural autocorrelation measure.

Conclusion on diffraction

From the lecture of B. Gelloz and from pages 66 and 67 of his slides, we have

Amplitude of scattering $F(k) = \int_{\text{solid}} \rho(x) e^{-ik \cdot x} dx$ (*)

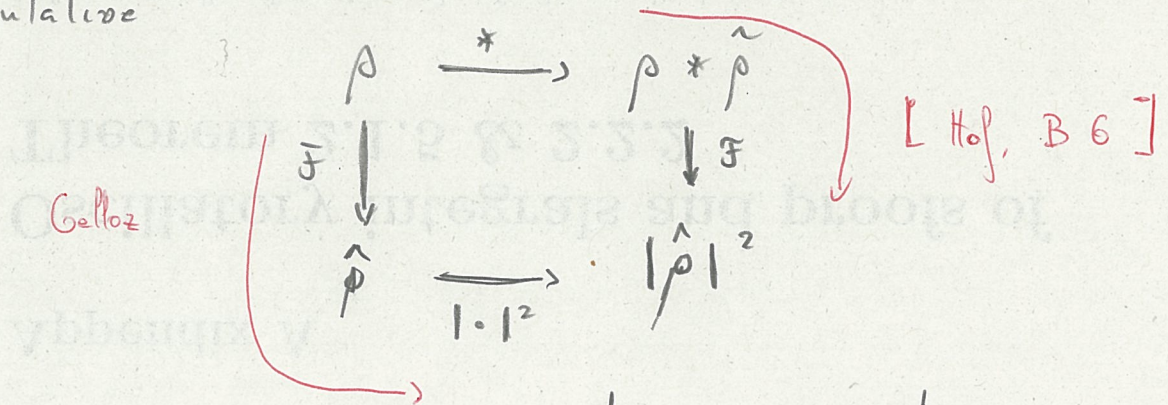
$$= \hat{\rho}(k)$$

with ρ the distribution of matter (electron).

The intensity I is then given by $I = |F|^2 = |\hat{\rho}|^2$.

Now is meaningful only if "solid" is small, but the relation $\rho(x+R) = \rho(x)$ implies that the solid is in the entire space. *Seems like a contradiction*

However, for ρ in L^1 the following diagram is commutative



We have used the second arrow to give a meaning to diffraction. Indeed, following [B 6], we considered

For any $\rho \in \mathcal{K}(\mathbb{R}^d)$ translation bounded

$\rho|_{B_R} * \tilde{\rho}|_{B_R}$ is well defined, translation bounded

and the net

$$\mathbb{R} \ni R \mapsto \frac{1}{\text{Vol}(B_R)} \rho|_{B_R} * \tilde{\rho}|_{B_R} \equiv \gamma_\rho^R$$

a net in the w^* compact set $K = \{ \gamma_\rho^R \}$ w^* -top.

Thus it has a least one converging subnet, and this limit is called the auto-correlation γ_ρ .

Then $I = |\hat{\gamma}_\rho|^2$ with the Fourier transform always well defined on $\mathcal{K}(\mathbb{R}^d)$, also in a suitable sense.

Note that the auto correlation might not be unique, but the theory allows it.