Large deviation

Consider $S_n := X_1 + X_2 + \cdots + X_n$

(a sum of independent, identically distributed r.v. with mean μ and variance σ^2)

By the law of large number $S_n \sim n\mu$ and of variance $n\sigma^2$, and deviation $:= \sqrt{var(S_n)} = \sqrt{n}\,\sigma$

It's unlikely that S_n will deviate from $n\mu$ by more than n^α $(\alpha > \frac{1}{2})$.

Such unlikely events are called large deviation.

Let X be a r.v. with $\mathbb{E}(X) = 0$,

and $M_X(t) = \mathbb{E}(e^{tX})$ exists for $|t| < \delta$ with $\delta > 0$

X is the common r.v. of identically distributed $X_1, X_2, \cdots$

Since $x \mapsto e^{tx}$ is strictly increasing for $t > 0$

$\Rightarrow S_n > na$ for $a \in \mathbb{R} \Leftrightarrow e^{tS_n} > e^{tna}$

$$\Rightarrow \mathbb{P}(S_n > na) = \mathbb{P}(e^{tS_n} > e^{tna}) \underset{\substack{\uparrow \\ \text{Markov's inequality}}}{\leq} \frac{\mathbb{E}(e^{tS_n})}{e^{tan}} \underset{\nwarrow \text{independence}}{=} \left(\frac{\mathbb{E}(e^{tX})}{e^{ta}}\right)^n = \left(\frac{M_X(t)}{e^{ta}}\right)^n$$

Since the left-hand side is independent of t, one has

$$\mathbb{P}(S_n > na) \leq \inf_{t \in \mathbb{R}^+} \left\{\left(\frac{M_X(t)}{e^{at}}\right)\right\}^n$$

Set $\Lambda(t) := \ln M_X(t)$

$$\frac{M_X(t)}{e^{ta}} = e^{-ta} e^{\Lambda(t)} = e^{-(at - \Lambda(t))}$$

$$\Rightarrow \inf_{t \geq 0} \left(\frac{M_X(t)}{e^{ta}}\right) = \inf_{t > 0} e^{-(at - \Lambda(t))} = e^{-\Lambda^*(a)}$$

with $\Lambda^*(a) = \sup\limits_{\substack{t > 0 \\ (t \in \mathbb{R})}} (at - \Lambda(t))$ (Fenchel-Legendre transform)

$\Rightarrow \mathbb{P}(S_n > an) \leq (e^{-\Lambda^*(a)})^n$

$\Leftrightarrow \log \mathbb{P}(S_n > an) \leq n \log e^{-\Lambda^*(a)} = -n\Lambda^*(a)$

$\Leftrightarrow \frac{1}{n} \log \mathbb{P}(S_n > an) \leq -\Lambda^*(a)$

Thm (Large deviation thm)

Let $X_1, X_2, \cdots$ be independent identically distributed random variables with mean 0 and common moment generating function M_X ~~de~~ defined on interval $(-\delta, \delta)$ for $\delta > 0$, let $a > 0$ such that $\mathbb{P}(X > a) > 0$

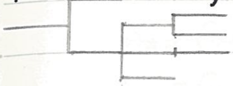
Then $\Lambda^*(a) > 0$ and

$\frac{1}{n} \log \mathbb{P}(S_n > na) \xrightarrow{n \to \infty} -\Lambda^*(a)$

Unprecisely, $\mathbb{P}(S_n > an)$ decays to 0 as $e^{-\Lambda^*(a) n}$

IX Branching process (discrete time reproduction process)

Each nomad lives for 1 unit of time
 and has k children with a probability p_k ($\sum_{k=0}^{\infty} p_k = 1$).
At time $\overset{n}{\cancel{k}}=0$, there's 1 nomad.
The number of children of each nomad ~~in~~ is independent.


(branching process)

Let Z_n denote the number of nomads at time n.
Clearly $\mathbb{P}(Z_0=1)=1$
 $\mathbb{P}(Z_1=k)=p_k$
 $\mathbb{P}(Z_2=k)$ already quite complicated
Let's denote by C the integer-valued random variable given by $\mathbb{P}(C=k)=p_k$.
And C_j will denote a r.v. with the same probability distribution
Then $Z_2 = C_1 + C_2 + \cdots + C_{Z_1}$
 $Z_n = C_1 + C_2 + \cdots + C_{Z_{n-1}}$
Recall that a random sum of random variables has been studied in Chapter IV
 It was treated with the probability generating function
 $G(s) := G_C(s) = \sum_{k=0}^{\infty} s^k p_k$
 $G_n(s) := G_{Z_n}(s) = \mathbb{E}(s^{Z_n}) = \sum_{k=0}^{\infty} s^k \overset{\mathbb{P}}{\cancel{\mathbb{E}}}(Z_n=k)$
Thm: For any $n \in \mathbb{N}^*$
* $G_0(s)=s$, $G_n(s)=G_{n-1}(G(s))$ and $G_n(s) = \underbrace{G(G(G\cdots(G}_{n \text{ times composition}}(s))\cdots))$ (*)

Proof: Since $Z_0=1 \Rightarrow G_0(s)=s$
 $G_n(s)=G_{n-1}(G(s))$ (shown in the random sum formula in Chapter IV)
 and by iteration we get (*)
Remark
 Knowing $G_n(s)$, we can compute $\mathbb{E}(Z_n=k)$ for any k

Let's set $\mu = \mathbb{E}(C) = \sum_{k=0}^{\infty} kp_k < \infty$

Thm $\mathbb{E}(Z_n) = \mu^n$

Proof: Recall $\mathbb{E}(Z_n) = G_n'(1) = G_{n-1}'(G(1))G'(1) = G_{n-1}'(1)\,\mathbb{E}(C) = \mu G_{n-1}'(1)$

by iteration, we get $\mathbb{E}(Z_n) = \mu^n$ $\square$

Exercise

If $\mu = \mathbb{E}(C)$ and $\sigma^2 = var(C)$

~~Then var(Zn) =~~

Then $var(Z_n) = \begin{cases} n^2\mu^{n-1}\frac{\mu^n - 1}{\mu - 1} & \text{if } \mu \neq 1 \\ n\sigma^2 & \text{if } \mu = 1 \end{cases}$

What about extinction?

Clearly if $\mu < 1$, then we have an asymptotic extinction.

But what about $\mu > 1$

Set $e := \mathbb{P}(Z_n = 0 \text{ for some } n \geq 0)$

Let $E_n := \{Z_n = 0\}$ the event that the branch process is extinct at the n^{th} generat

and set $e_n = \mathbb{P}(E_n)$

Set $\bigcup_{n=1}^{\infty} E_n = \{\exists n \in \mathbb{N}^* : Z_n = 0\}$ and observe that $E_n \subset E_{n+1} \Rightarrow e_n \leq e_{n+1}$

Since this sequence is increasing and since the probability measure is continuous,

we get $e = \lim_{n\to\infty} e_n$

Remark: If $P_0 = 0$ then $e = 0$

Thm (Extinction probability thm)

The probability of e is given by the smallest non-negative root of the e

$x = G(x)$

Proof: Recall $e_n = \mathbb{P}(Z_n = 0) = G_n(0)$

Since $G_n(s) = G \circ G \circ \cdots \circ G(s) = G(G_{n-1}(s))$

We infer $e_n = G_n(0) = G(G_{n-1}(0)) = G(e_{n-1})$

for any $n = 1, 2, \cdots$, with the initial condition $e_0 = 0$

Taking $\lim_{n\to\infty}$ on both sides

$e = \lim_{n\to\infty} e_n = \lim_{n\to\infty} G(e_{n-1}) = G(\lim_{n\to\infty} e_{n-1}) = G(e)$ since G is continuous on [

To show it's the smallest one, we assume $\eta \in (0,1)$ satisfying $\eta = G(\eta)$.

Since $G \nearrow$ on $(0,1)$ (since $G'(s) = \sum_{k=0}^{\infty} k s^{k-1} p^k \geq 0$)

One has

$$e_1 = G_1(0) = G(0) \le G(\eta) = \eta$$

$\cancel{e_{n-1} = G(e_1) \le G(\eta) = \eta}$ Assume $e_{n-1} \le \eta$,

$e_n = G(e_{n-1}) \le G(\eta) = \eta$ $\therefore$ By iteration,

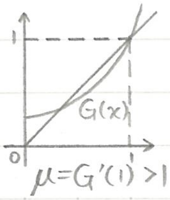
$e = \lim_{n\to\infty} e_n \le \eta$. □

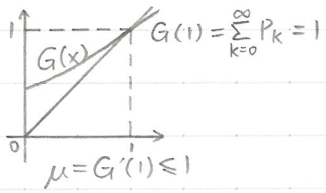
Thm $P_1 \neq 1$ $\overset{\text{then}}{\cancel{\Rightarrow}}$ $(e = 1 \text{ iff } \mu \le 1)$

Proof: Suppose $P_0 > 0$

Then on $[0,1]$, G_n is continuous, increasing and convex. ($G_n'' > 0$)

Only 2 situations appear:





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