Reminder

$(\Omega, \mathcal{F}, \mathbb{P})$ a probability space ; $X: \Omega \to \mathbb{R}$ a random variable ;

$F_X : \mathbb{R} \to [0,1]$ its distribution function, $F_X(x) = \mathbb{P}(X \leq x)$;

$\mathbb{E}(X^k)$ k-moment ; $var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$; $cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

$M_X : I \to \mathbb{R}$ moment generating function, $M_X(t) = \mathbb{E}(e^{tX})$ if absolutely convergent

Markov + Jensen's inequality

Def. The characteristic function ϕ_X of a r.v. X

$$\phi_X : \mathbb{R} \to \mathbb{C},\ \phi_X(t) = \mathbb{E}(e^{itX}) = \mathbb{E}(\cos(tX)) + i\,\mathbb{E}(\sin(tX))$$

Note that $\mathbb{E}(|e^{itX}|) = \mathbb{E}(1) = \lim_{x \to \infty} F_X(x) = 1$

$\therefore$ ϕ_X always exists since the corresponding integral is always absolutely convergent.

Remark

If $M_X(t)$ exists for $|t| \leq R$, then $\phi_X(t) = M_X(it)$

But ϕ_X exists even if M_X does not exist.

E.g. For the Cauchy distribution, $\phi_X(t) = e^{-|t|}$

Thm. If $\mathbb{E}(X^k)$ exists for all $k \leq N < \infty$, then

$$\phi_X(t) = \sum_{k=0}^{N} \frac{1}{k!} (it)^k \, \mathbb{E}(X^k) + o(t^N)$$

Note: $f \in o(t^N)$ if $\lim_{t \to 0} \frac{f(t)}{t^N} = 0$; $f \in O(t^N)$ if $\frac{f(t)}{t^N}$ is bounded near 0 (zero).

Properties

1) If $a, b \in \mathbb{R}$, $\phi_{aX+b}(t) = e^{itb} \phi_X(at)$

2) If X, Y independent, $\phi_{X+Y}(t) = \phi_X(t) \cdot \phi_Y(t)$

3) ϕ_X determines uniquely F_X, but the link is always simple.

However if X is absolutely continuous, then

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_X(t)\, dt$$

VIII The main limit theorem

⚠ When we say sth converges, we have to say in which sense it converges

- Law of large numbers

If we throw a die many times, the average will converge to 3.5

Def. A sequence ~~z_1~~, $Z_1, Z_2, \cdots$ of random variables converges in mean square to the r.v. Z_∞ if $\mathbb{E}((Z_n - Z_\infty)^2) \xrightarrow{n\to\infty} 0$

We write $Z_n \xrightarrow{n\to\infty} Z_\infty$ in mean square.

Thm. Mean square law of large numbers

Let $X_1, X_2, \cdots$ a sequence of independent r.v., ~~e~~each with mean μ and variance σ^2 (same), then the average of the first n of X_i satisfies

$$\frac{1}{n}(X_1 + X_2 + \cdots + X_n) \xrightarrow{n\to\infty} \mu \text{ in mean square}$$

Proof:

Set $S_n = \sum_{i=1}^{n} X_i$, then

$$\mathbb{E}\left(\frac{1}{n}S_n\right) = \frac{1}{n} n\mu = \mu \text{ and}$$

convergence in mean square

$$\mathbb{E}\left(\left(\frac{1}{n}S_n - \mu\right)^2\right) = \mathrm{var}\left(\frac{1}{n}S_n\right) = \frac{1}{n^2}\mathrm{var}(S_n) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n} \xrightarrow{n\to\infty} 0 \qquad \square$$

because of independence

Remark

If X is a r.v. with $\mathbb{E}(X) = \mu$ and $\mathrm{var}(X) = 0$, then $X = \mu$.

Another type of convergence

Def. Let $Z_1, Z_2, \cdots$ be a sequence of r.v. It converges in probability if

if $\forall \varepsilon > 0$, $\mathbb{P}(|Z_n - Z_\infty| \geq \varepsilon) \xrightarrow{n\to\infty} 0$

We write $Z_n \xrightarrow{n\to\infty} Z_\infty$ in probability.

Proposition

If $Z_n \xrightarrow{n\to\infty} Z_\infty$ in mean square, then $Z_n \xrightarrow{n\to\infty} Z_\infty$ in probability.

(stronger) (weeker)

⚠ The converse is not true.

Lemma (Chebyshev's inequality)

If X is a r.v. with $\mathbb{E}(X^2)<\infty$,
then $\mathbb{P}(|X|\geqslant x)\leqslant\frac{\mathbb{E}(X^2)}{x^2}$ for any $x\in\mathbb{R}_+^*$

Proof:

By Markov's inequality, to X^2, ~~$(X^2\geqslant 0)$~~
$\forall x\in\mathbb{R}_+^*, x^2\in\mathbb{R}_+^*$, $\mathbb{P}(|X|\geqslant x)=\mathbb{P}(X^2\geqslant x^2)\leqslant\frac{\mathbb{E}(X^2)}{x^2}$ □

Proof of proposition

Consider $X:=Z_n-Z_\infty$, ~~then~~ and any $\varepsilon>0$, then
$\mathbb{P}(|Z_n-Z_\infty|>\varepsilon)\leqslant\mathbb{P}(|Z_n-Z_\infty|\leqslant\varepsilon)\leqslant\frac{\mathbb{E}((Z_n-Z_\infty)^2)}{\varepsilon^2}$

$\because\mathbb{P}(|Z_n-Z_\infty|>\varepsilon)\geqslant 0$ ~~and~~

$\therefore Z_n\xrightarrow{n\to\infty}$ ~~in~~ Z_∞ in mean square $\Leftrightarrow\mathbb{E}((Z_n-Z_\infty)^2)\xrightarrow{n\to\infty}0$

$$\Downarrow$$

$Z_n\xrightarrow{n\to\infty}Z_\infty$ in probability $\Leftrightarrow\mathbb{P}(|Z_n-Z_\infty|>\varepsilon)\xrightarrow{n\to\infty}0$

Exercise: Let Z_n be discrete r.v. for $n\in\mathbb{N}_+$

Consider $\mathbb{P}(Z_n=0)=1-\frac{1}{n}$ and $\mathbb{P}(Z_n=n)=\frac{1}{n}$

Show that $Z_n\xrightarrow{n\to\infty}0$ in probability but not in mean square.

Remark. Chebyshev's inequality also reads

$$\mathbb{P}(|X-\mathbb{E}(X)| \geq x) \leq \frac{var(X)}{x^2}$$

Central limit thm's motivation

If $X_1, X_2, \cdots$ are independent and identically distributed r.v. ~~wi~~with mean μ and variance $\sigma^2 \neq 0$

Then $S_n = X_1 + \cdots + X_n \sim n\mu$

But what about the difference $S_n - n\mu$?

We shall work with standardized version of S_n:

$$Z_n := \frac{S_n - \mathbb{E}(S_n)}{\sqrt{var(S_n)}} = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

Thm (Central limit theorem)

Let $X_1, X_2, \cdots$ be independent and identically distributed r.v. with mean μ and variance $\sigma^2 \neq 0$

Let $S_n = \sum_{j=1}^{n} X_j$ and $Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$

Then $\forall x \in \mathbb{R}$, $\mathbb{P}(Z_n \leq x) \xrightarrow{n\to\infty} \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$ (f_X for normal distributio with pareameter $(0,1)$)

Def. A sequence $Z_1, Z_2, \cdots$ of r.v. ~~conv~~ converges in distribution (or weekly) ~~if~~ to Z_∞ ~~if~~(we write $Z_n \xRightarrow{n\to\infty} Z_\infty$) if $\mathbb{P}(Z_n \leq x) \xrightarrow{n\to\infty} \mathbb{P}(Z_\infty \leq x)$ $\forall x \in C$,

$$C = \{x \in \mathbb{R} \mid F_{Z_\infty} \text{ is continuous at } x\}$$

Remark: The central limit thm is a week convergence since the normal distribution has no point of discontinuity.

Remark: If $Z_n \xrightarrow{n\to\infty} Z_\infty$ in probability, then $Z_n \xRightarrow{n\to\infty} Z_\infty$

The converse ~~if~~is false in general but not always.

But if Z_∞ is constant then $Z_n \xrightarrow{n\to\infty} Z_\infty$ in probability $\Leftrightarrow Z_n \xRightarrow{n\to\infty} Z_\infty$

Thm. (Continuity theorem)

Let $Z_1, Z_2, \cdots$ be a sequence of random variables with characteristic ϕ_n

Then $Z_n \xRightarrow{n\to\infty} Z_\infty$ iff $\phi_n(t) \xrightarrow{n\to\infty} \phi_\infty(t)$ for all $t \in \mathbb{R}$

(pointwise convergence)

New version of the law of large numbers

Let $X_1, X_2, \cdots$ be independent r.v. which are identically distributed and with mean μ (variance could not exist) ($|\mu| < \infty$)

Then $\frac{1}{n} S_n \xrightarrow{n\to\infty} \mu$ in probability

Indeed, set $U_n = \frac{1}{n} S_n$, then

$$\phi_{U_n}(t) = \left(\phi\left(\tfrac{t}{n}\right)\right)^n = \left(1 + i\tfrac{t}{n}\mu + o(t)\right)^n \xrightarrow{n\to\infty} e^{it\mu} = \phi_\mu(t)$$

$$\Rightarrow \phi_{U_n}(t) \xrightarrow{n\to\infty} \phi_\mu(t) \text{ pointwise } (\forall t)$$

$$\Leftrightarrow U_n \xRightarrow{n\to\infty} \mu \text{ (constant)} \Leftrightarrow U_n \xrightarrow{n\to\infty} \mu \text{ in probability.}$$ □

Proof of central limit thm

Recall $Z_n = \frac{S_n - \mu n}{\sigma \sqrt{n}} = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{n} \underbrace{(X_i - \mu)}_{:= U_i}$ with $\mathbb{E}(U_i) = 0$

$\mathbb{E}(U_i^2) = var(U_i) = var(X_i - \mu) = var(X_i) = \sigma^2$

Then $\phi_{Z_n}(t) = \mathbb{E}(e^{itZ_n}) = \mathbb{E}\left(\exp\left(it\frac{1}{\sigma\sqrt{n}}\sum_{j=1}^{n} U_j\right)\right)$

$$= \mathbb{E}\left(\exp\left(it\frac{1}{\sigma\sqrt{n}} U\right)\right)^n = \phi_U\left(\frac{t}{\sigma\sqrt{n}}\right)^n = \left(1 + \frac{1}{2}\left(i\frac{t}{\sigma\sqrt{n}}\right)^2 \sigma^2 + o(t^2)\right)^n$$

$$= \left(1 - \frac{t^2}{2n} + o(t^2)\right)^n \xrightarrow{n\to\infty} e^{-\frac{1}{2}t^2} = \phi_{N(0,1)}$$ normal distribution

$$\Rightarrow \phi_{Z_n}(t) \xrightarrow{n\to\infty} \phi_{N(0,1)}(t) \ \forall t$$

$$\Leftrightarrow Z_n \xrightarrow{n\to\infty} N(0,1) \text{ in probability}$$ □