XIII Continuous time Markov chain

Consider $\underline{X} = \{X(t) \mid t \geq 0\}$, with $X(t) : \Omega \to S$ a r.v.

state space countable

sample space

Def. X is a continuous time Markov chain if it satifies the Markov condition

$\forall t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1}$:

$$\mathbb{P}(X(t_{n+1}) = i_{n+1} \mid X(t_0) = i_0, X(t_1) = i_1, \cdots, X(t_n) = i_n) = \mathbb{P}(X(t_{n+1}) = i_{n+1} \mid X(t_n) = i_n)$$

⚠ There's no analogue for the transition matrix P, ∵ because there's no natural unit time.

Def. $\forall i, j \in S$ and $s \leq t$ we set (the transision probability)

$P_{i,j}(s,t) := \mathbb{P}(X(t) = j \mid X(s) = i)$

If homogeneous $\mathbb{P}(X(t-s) = j \mid X(0) = i) =: P_{i,j}(t-s)$

Prop. If homogeneous, $\{P(t) \equiv P_t \equiv (P(t))_{i,j} \mid t \geq 0\}$ is a semi-group. Indeed one has

1) $P_0 = 1$

2) $P_{i,j}(t) \geq 0$, $\sum_j P_{i,j}(t) = 1$ (stochastic property)

3) $P_{s+t} = P_s P_t$ (Chapman-Holmogorov)

Def. The semi-group is standard if $\lim_{t \to 0} P_{i,j}(t) = \delta_{i,j}$ for any i, j (continuity)

Remark: This continuity at 0 implies the same continuity at any $t \geq 0$, since (3)

Thm. Let $\{P_t \mid t > 0\}$ be a standard stochastic semi-group.

Then $\exists G = \{g_{i,j}\}$:

1) $P_{i,j}(t) = g_{i,j} t + o(t) \quad \forall i \neq j$

2) $P_{i,i}(t) = 1 + g_{i,i}(t) + o(t) \quad \forall i$

3) $0 \leq g_{i,j} < \infty$ if $i \neq j$

$0 \geq g_{i,i} \geq -\infty$

Remark: Think about $f : t \to 1 - \sqrt{t}$, then if we take the Taylor's expansion

$P_1 = 1 - \infty t$

The matrix G is called the generater of $\{P_t\}$

Formally, one has $G = \lim_{t \to 0} \frac{1}{t}(P_t - 1)$

Def. The semi-group is uniform if $P_{i,i}(t) \xrightarrow{t \to 0} 1$ uniformly in i.

$(\Rightarrow P_{i,j}(t) \xrightarrow{t \to 0} 0)$

Thm. $\{P_t\}$ is uniform iff $\sup_i\{-g_{i,i}\}<\infty$

Thm. Let $\{P_t\}$ be a uniform stochastic semi-group with generator G.

Then it's the unique solution to

- forward equation $P_t' = P_t G$
- backward equation $P_t' = G P_t$

with $P_0 = 1$.

Remark: a big difference with discrete time:

- If $\{P_t\}$ is standard, $\forall t \geq 0: P_{i,i}(t) > 0$
- Either $\forall t > 0: P_{i,j}(t) > 0$; or $\forall t > 0: P_{i,j}(t) = 0$ (Lévy dichotomy)

Def. X is irreducible if $\forall i,j \in S, \exists t > 0: P_{i,j}(t) > 0$

Def. Vector $\pi = (\pi_i)_{i \in S}$ is a stationary distribution if

$$\pi_i \geq 0, \sum_{i \in S} \pi_i = 1 \ \&\& \ \pi = \pi P_t \ \forall t \geq 0 \quad (\pi \text{ is a row vector})$$

Remark: If $\{P_t\}$ is uniform and $\pi = \pi P_t$ then

$$0 = \pi P_t' = \pi P_t G = \pi G \Rightarrow \pi \in \ker G$$

Thm. Let X be irreducible with transision semi-group P_t

1) If $\exists \pi$ stationary distribution, then π is unique

ant $P_{i,j}(t) \to \pi_j$ as $t \to \infty$

2) If $\neg \exists \pi$, then $P_{i,j}(t) \to 0$ as $t \to \infty$ $\forall i,j$ (spread on all possible states)

Remark. This implies if the states are finite then $\exists \pi$, ~~or 2) will cau:~~

or 2) will cause a contradictory

Wiener Process ≡ Brownian Motion

Recall the random work 1D

S_n = posision of particle at time n

$$S_n = \begin{cases} S_{n-1} + 1 & \text{with prob. } p \\ S_{n-1} - 1 & \text{with prob. } q \end{cases}$$

$S_n = S_0 + X_1 + X_2 + \cdots + X_n$ with X_j i.i.d.r.v.

Properties

1) Time homogeneity:

$S_m - S_0$ and $S_{m+n} - S_n$ have the same distribution.

2) Independent increment:

$S_{n_1} - S_{m_1}$ and $S_{n_2} - S_{m_2}$ are indep. if $(m_1, n_1] \cap (m_2, n_2] = \phi$

Let $\underline{X} = \{X(t) \mid t \geq 0\}$ with $X(t): \Omega \to \mathbb{R}$ (1D Wiener process)

Def. For fixed $\omega \in \Omega$, the set $\{\underbrace{X(t,\omega)}_{\text{or } [X(t)](\omega), \text{ is a function of } t} \mid t \geq 0\}$ is called a sample path.

Def. X is called a Gaussian process if

$\forall t = (t_1, t_2, \cdots, t_n)$ the family $(X(t_1), \cdots, X(t_n))$ has the multivariate normal distribution $N(\mu(t), X(t))$ with mean vector $\mu(t)$ and covariant matrix $V(t)$. $\mu_i(t) = \mathbb{E}(X(t_i))$

$V_{i,j} = \mathbb{E}((X_i - \mathbb{E}(X_i))(X_j - \mathbb{E}(X_j)))$

Def. Wiener process $\underline{W} = \{W(t) \mid t \geq 0\}$ starting at $W(0) = w \in \mathbb{R}$ is a Gaussian process such that

1) $\underline{W}$ has independent increment

2) $W(s+t) - W(s)$ is distributed as $N(0, \sigma^2 t)$ with $\forall s, t \geq 0$ and fixed $\sigma^2 \in \mathbb{R}$

3) The sample paths of $\underline{W}$ are continuous

- $\underline{W}$ is called standard if $w = 0$, $\sigma^2 = 1$
- $\underline{W}$ has Markov property, namely $\forall t_1 < t_2 < t_3 < \cdots < t_n$,

$$\mathbb{P}(X(t_n) \leq x \mid X(t_1) = x_1, \cdots, X(t_{n-1}) = x_{n-1}) = \mathbb{P}(X(t_n) \leq x \mid X(t_{n-1}) = x_{n-1})$$

- $\mathrm{cov}(W(s), W(t)) = \sigma^2 \min\{s, t\}$ (auto covariance)

Suppose $\underline{W}$ is standard and $W(s)=x\in\mathbb{R}$

Then $W(t)$ is distributed as $N(x, t-s)$ for $t\geq s$

i.e. $F(t,y|s,x) \doteq \mathbb{P}(W(t)\leq y \mid W(s)\leq x)$ has density

$$f(t,y|s,x)=\frac{1}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{(y-x)^2}{2(t-s)}\right)$$

Observe that f is solution of

$$\begin{cases} \dfrac{\partial f}{\partial t}=\dfrac{1}{2}\dfrac{\partial^2 f}{\partial y^2} & \left(\begin{array}{c}\text{Forward}\\ \text{diff. eq.}\end{array}\right) \\[2ex] \dfrac{\partial f}{\partial s}=-\dfrac{1}{2}\dfrac{\partial^2 f}{\partial x^2} & \left(\begin{array}{c}\text{Backward}\\ \text{diff. eq.}\end{array}\right) \\[2ex] + \text{boundary conditions} \end{cases}$$

Question: Can we find a process for more general equations?

Let $\underline{D}=\{D(t)\mid t\geq 0\}$ be a process satisfying

1) Continuous sample path
2) $\mathbb{P}(|D(t+h)-D(t)|>\varepsilon \mid D(t)=x)=o(h)\ \forall\varepsilon>0$
3) $\mathbb{E}(D(t+h)-D(t) \mid D(t)=x)=a(t,x)h+o(h)$
4) $\mathbb{E}([D(t+h)-D(t)]^2 \mid D(t)=x)=b(t,x)h+o(h)$

with a = instantaneus drift, b = instantaneous variance

(a,b are given by the physics)

If we set $f(t,y|s,x)=D_y\,\mathbb{P}(D(t)\leq y \mid D(s)=x)$ then f satisfies

$$\begin{cases} \dfrac{\partial f}{\partial t}=-\dfrac{\partial f}{\partial y}(a(t,y)f)+\dfrac{1}{2}\dfrac{\partial^2}{\partial y^2}(b(t,y)f) & \text{(f.e.)} \\[2ex] \dfrac{\partial f}{\partial s}=-a(s,x)\dfrac{\partial f}{\partial x}-\dfrac{1}{2}b(s,x)\dfrac{\partial^2 f}{\partial x^2} & \text{(b.e.)} \end{cases}$$

Wiener process: $a=0,\ b=\sigma^2$ ($=1$ if normal)

Wiener process with drift: $a=m,\ b=\sigma^2$

Orstein-Uhlenbeck process: $a(t,x)=-\beta x,\ b=\sigma^2$