No.

Date

For X a hom. M.C. we set

$T_j = \min_{n\geq 1} \{X_n = j\}$ (for a fixed j, the first time to reach j) (first passage time)

$f_{i,j}(n) := \mathbb{P}(T_j = n \mid X_0 = i)$ (first passage probability)

$=: \mathbb{P}_i(T_j = n)$

Def. A state $i \in S$ is recurrent if $\mathbb{P}_i(T_i < \infty) = 1$ and is transient otherwise.

Thm. The state i is recurrent iff $\sum_{n\in\mathbb{N}} p_{i,i}(n) = \infty$ $(*)$

Remark: We proved sth similar in Chapter 9 with 2 generating functions.

Pf. Let $i,j \in S$, we set

$P_{i,j}(s) := \sum_{n\in\mathbb{N}} p_{i,j}(n) s^n$ with $p_{i,j}(0) = \delta_{i,j}$

$F_{i,j}(s) := \sum_{n\in\mathbb{N}} f_{i,j}(n) s^n$ with $f_{i,j}(0) = 0$

Observe that

$f_{i,j} := F_{i,j}(1) = \sum_{n\in\mathbb{N}} f_{i,j}(n) = \sum_{n\in\mathbb{N}} \mathbb{P}_i(T_j = n) = \mathbb{P}_i(T_j < \infty)$

So i is recurrent iff $f_{i,i} = 1$. $\square$

Thm.

- In a communicating class, all states are either recurrent or transient at the same time.
- If S is finite, $\exists$ recurrent state. (quite natural)

Lemma: $\forall\, i,j \in S$, and $s \in (-1,1]$, one has

$$P_{i,j}(s) = \delta_{i,j} + F_{i,j}(s) P_{j,j}(s)$$

Proof of $(*)$:

If $i=j$, $P_{i,i}(s) = 1 + F_{i,i}(s) P_{i,j}(s) \Leftrightarrow P_{i,i}(s) = \dfrac{1}{1-F_{i,i}(s)}$ ♡

$\lim_{s\nearrow 1} F_{i,i}(s) = f_{i,i} = \mathbb{P}_i(T_i < \infty)$

($f_{i,i} = 1$ iff i is recurrent)

$\lim_{s\nearrow 1} P_{i,i}(s) = \sum_{n\in\mathbb{N}} P_{i,i}(n)$, and with ♡ we finish the proof. $\square$

Random walk on $\mathbb{Z}^d$

Consider $\mathbb{Z}^d \ni x, y$, the symmetric walk is a Markov chain with $S = \mathbb{Z}^d$

$$p_{x,y} = \begin{cases} \frac{1}{2d} & \text{if } x \sim y \\ 0 & \text{otherwise} \end{cases} \qquad x \sim y \Leftrightarrow \text{they have a common edge} \Leftrightarrow |x-y|_1 := \sum_{n=1}^{d} |(x-y)_n| = 1$$

Thm. Polya' thm

For $d = 1$ or 2, any state is recurrent

For $d \geqslant 3$, any state is transient (lost in dimension)

A generalization of first time passage

Def. For any $A \subset S$, the hitting time H^A is defined by

$$H^A = \inf \{ n \geqslant 0 \mid X_n \in A \}$$

Remark: H^A takes values $\{0, \cdots, \infty\}$ $\left(\inf \phi \overset{\text{by def}}{=\!=} \infty\right)$

The hitting probability is the prob. of reaching A starting at i in a finite time.

$$h_i^A := \mathbb{P}_i (H^A < \infty)$$

Thm. Set $h^A = (h_i^A)_{i \in S}$, then h^A is the minimal non-negative solution of

$$h_i^A = \begin{cases} i & \text{if } i \in A \\ (P h^A)_i & \text{if } i \notin A \end{cases} \qquad (*)$$

with h^A a vector; "minimal" means if $X = (X_i)_{i \in S}$ is another solution then $\forall i \in S : h_i^A \leqslant X_i$

Def. The mean hitting times $K_i^A := \mathbb{E}_i (H^A)$

with the convention $K_i^A = \infty$ if $\mathbb{P}(H^A = \infty) > 0$

Thm. If $K^A = (K_i^A)_{i \in S}$ then K^A is the minimal solution of

$$K_i^A = \begin{cases} 0 & \text{if } i \in A \\ 1 + (P K^A)_i & \text{if } i \notin A \end{cases}$$

Classification of states

Thm. Suppose $X_0 = i$ and define $V_i = |\{ n \geqslant 1 \mid X_n = i \}| \in \mathbb{N} \cup \{\infty\}$

(the number of subsequent visits to i)

Then V_i has a geometric distribution

$$\mathbb{P}_i (V_i = r) = (1 - f_{i,i}) f_{i,i}^r$$

with $f_{i,i} = \mathbb{P}_i (T_i < \infty) = \mathbb{P}_i (X_n = i \text{ for some } n)$

In particular, $\mathbb{P}_i (V_i = \infty) = 1$ if $f_{i,i} = 1$ ($\Leftrightarrow$ i is recurrent)

$\mathbb{P}_i (V_i < \infty) = 1$ if i is transient

Def.

1) Mean recurrence time μ_i for the state i is

$$\mu_i = \mathbb{E}_i(T_i) = \sum_{n\in\mathbb{N}} n f_{i,i}(n) = \sum_{n\in\mathbb{N}} n \mathbb{P}_i(T_i = n) \text{ if } i \text{ is recurrent}$$

$$\hookrightarrow = \infty \qquad \text{if } i \text{ is transient}$$

2) If i is recurrent and $\mu_i = \infty$, then i is null;

$\hookrightarrow \mu_i < \infty$, then i is positive (or non-null)

3) The period d_i for the state i is defind by

aperiodic 2, 4, 6, ⋯ 3, 6, 9, ⋯ also 5, 7, ⋯

$d_i = \gcd\{n \mid p_{i,i}(n) > 0\}$ with gcd: greatest common divisor 最大公因数

If $d_i = 1$, the state is aperiodic, while if $d_i > 1$ then it's periodic.

4) If i is recurrent, positive and aperiodic, then i is called engodic.

Thm. These properties are shared in any communicating class.

Corollary: If the chain is irreducible, and if one (and then all) state is recurrent, then $\mathbb{P}(X_n = j \text{ for some } n \geq 1) = 1$

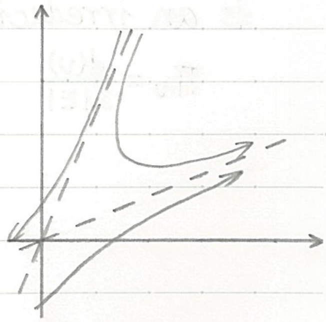
independently of the initial state.

Def. Let X be a Markov chain, with transition matrix P

A vector $\pi = (\pi_i)_{i\in S}$ is an invariant distribution if
(stationary/equillibrium)

1) $\pi_j \geq 0$ and $\sum_{i\in S} \pi_i = 1$

2) $\pi = \pi P \Rightarrow \pi = \pi P^n$ (regard π as a line vector)



Thm. If $\underline{X}$ is irreducible, $\exists$ invariant distribution π iff all states are positive recurrent, and then

$$\forall i \in S: \pi_i = \frac{1}{\mu_i}$$

The invariant distribution is unique.

Thm. (convergence to equillibrium)

Suppose X is irreducible, each state is engodic, then

$$\forall i,j \in S: p_{i,j}(n) \xrightarrow{n\to\infty} \pi_j$$

Def. A stochastic matrix $P=(p_{i,j})_{i,j\in S}$ and a distribution $\lambda=(\lambda_i)_{i\in S}$ is in detailed balance if

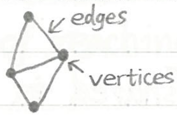
$$\forall i,j\in S: \lambda_i p_{i,j} = \lambda_j p_{j,i} \qquad (**$$

An irreducible M.c. $\underline{X}$ with transision matrix P and invariance distribution ~~π~~ Π is reversible ~~if~~ in equillibrium if

$$\forall i,j\in S: \Pi_i p_{i,j} = \Pi_j p_{j,i} \qquad (*$$

Remark

$\Pi=\Pi P$ is a global equillibrium; on the other hand, $(*)$ is a local equillibrium, the flow in 1 direction is equal to the flow in the other direction. In addition $(**)$ for $\forall i,j$ then $\lambda=\Pi$ and $\Pi=\Pi P$.



Random walk on finite graphs

A graph $G=(V,E)$ with $u,v\in V$, then $u\sim v$ iff $\exists e=(u,v)\in E$

The ~~di~~ degree of $u\in V$ $d(u)=|\{v\sim u\}|$

If G is finite then one has $\sum_u d(u)=2|E|$

Thm.

A random walk on a finite connected graph $G=(V,E)$ is an irreducible Markov chain with unique invarient distribution

$$\Pi_v=\frac{d(v)}{2|E|}$$