

XII Markov Chains

Main idea: the future depends on the present time but not the past

Remark

Consider a prob. space $(\Omega, \mathcal{F}, \mathbb{P})$ and random variables on it taking values in a countable set $S = \{i_j\}_{j \in \mathbb{N}}$ (state space)

i.e. $X_n : \Omega \rightarrow S$ s.t. $X_n^{-1}(i_j) \in \mathcal{F}$

\uparrow
index for a sequence of r.v.

Def. A sequence $X = (X_n)_{n \in \mathbb{N}}$ of r.v. is called a Markov chain if it has the Markov property, namely:

$$\mathbb{P}(X_{n+1} = i_{n+1} | X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = \mathbb{P}(X_{n+1} = i_{n+1} | X_n = i_n) \quad \text{for all } n \in \mathbb{N}.$$

X is homogeneous if

$$\mathbb{P}(X_{n+1} = j | X_n = i) = \mathbb{P}(X_1 = j | X_0 = i) \text{ for any } n \in \mathbb{N}. \quad (\text{independence in } n)$$

Examples: Branching process, Chapter 9

1D random walk, Chapter 10

Poisson process, Chapter 11

For X a homogeneous Markov chain, we define

- transition matrix $P = (p_{i,j})_{i,j \in S}$

with $p_{i,j} = \mathbb{P}(X_1 = j | X_0 = i)$ (from i to j)

- initial distribution $\lambda = (\lambda_j)_{j \in S}$

with $\lambda_j = \mathbb{P}(X_0 = j)$

Lemma

1) $\lambda_j \geq 0, \sum_{j \in S} \lambda_j = 1$ (probability mass distribution function)

2) P is a stochastic matrix, namely

$$p_{i,j} \geq 0 \text{ and } \sum_j p_{i,j} = 1$$

Thm: Let $\lambda = (\lambda_j)_{j \in S}$ be a initial distribution,

and let P be a stochastic matrix.

Markov

Then a sequence $X = (X_n)_{n \in \mathbb{N}}$ is a homogeneous chain with initial distribution λ and transition matrix P iff

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_n = i_n) = \lambda_{i_0} P \lambda_{i_1} p_{i_1, i_2} \dots p_{i_{n-1}, i_n} \quad (*)$$

An apparently stronger property:

Thm (extended Markov property)

Let X be a Markov chain, and fix $n > 0$

Call H any event depending on X_0, \dots, X_{n-1}

" " " " " " " " X_{n+1}, \dots

Then $P(F | X_n = i, H) = P(F | X_n = i)$

idea of the proof:

$$P(F | X_n = i, H) = \frac{P(F, X_n = i, H)}{P(X_n = i, H)},$$

then substitute (*) into it, and everything about H will be cancelled.

Def. Let X be a homogeneous Markov chain and $n \in \mathbb{N}^*$. We set

$p_{i,j}(n) = P(X_n = j | X_0 = i)$ which consist a matrix $P(n)$

We call it the n -step transition matrix.

$$\therefore p_{i,j}(1) = p_{i,j}$$

$$\text{We also define } p_{i,j}(0) = \delta_{i,j} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

Thm. (Chopman-Kolmogozov equation)

$$\forall n, m \in \mathbb{N} : P(n+m) = P(n)P(m)$$

$$\Leftrightarrow p_{ij}(n+m) = \sum_{k \in S} p_{ik}(m) p_{kj}(n)$$

Proof. (for $m > 0$)

$$p_{ij}(n+m) = P(X_{m+n} = j | X_0 = i) = \sum_{k \in S} P(X_{m+n} = j | X_m = k, X_0 = i) P(X_m = k | X_0 = i)$$

$$= \sum_{k \in S} P(X_{m+n} = j | X_m = k) \quad P(X_m = k | X_0 = i)$$

$$= \sum_{k \in S} P(X_n = j | X_0 = k) \quad P(X_m = k | X_0 = i)$$

$$= \sum_{k \in S} p_{ik}(m) p_{kj}(n)$$

And apparent when

$$m=0.$$

Remark: if P is a stochastic matrix, of size $N \times N$,

then 1 is always an eigenvalue.

$$\text{Indeed, } P \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \text{ since } \sum_j p_{ij} = 1$$

Def

Let $i, j \in S$; \times a hom. M.C. We say that i communicate with j .
(and write $i \leftrightarrow j$) if

$$\exists m, n \geq 0 : p_{i,j}(n) > 0 \text{ and } p_{j,i}(m) > 0$$

Lemma: \leftrightarrow is an equivalence relation in S , which means:

- reflexive: $\forall i \in S, i \leftrightarrow i$

- symmetric: $\forall i, j \in S, i \leftrightarrow j \Rightarrow j \leftrightarrow i$

- transitive: $\forall i, j, k \in S, i \leftrightarrow j \text{ and } j \leftrightarrow k \Rightarrow i \leftrightarrow k$

$\Rightarrow S$ can be divided into equivalence class, which means

C is an equivalence class if $\forall i, j \in C : i \leftrightarrow j$

We write $S = \bigsqcup C_i$.

$$(\Leftrightarrow S = \bigsqcup C_i \text{ and } \forall i \neq j : C_i \cap C_j = \emptyset)$$

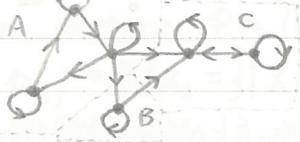
We say that \times is irreducible if S consists only 1 equivalence class.

We also write $i \rightarrow j$ if $\exists n : p_{i,j}(n) > 0$

A subset $C \subseteq S$ is closed if $\forall i \in C : i \rightarrow j \Rightarrow j \in C$

If a singleton $\{i\}$ is closed, it is called an absorbing state.

a class with only one element



3 communicating classes A, B, C

Lemma: A subset $C \subseteq S$ is closed iff

$$\forall i \in C, j \notin C : p_{ij} = 0 \quad (*)$$

Proof: \Rightarrow : If $p_{ij} \neq 0$, the C is not closed.

\Leftarrow : Assume (*). Let $k \in C$ and $l \in S$ such that $k \rightarrow l$.

$$\therefore \exists m \geq 0 : p_{k,l}(m) = P(X_m = k | X_0 = l) > 0$$

Then \exists a sequence $k_0 = k, k_1, k_2, \dots, k_m = l$ such that

$$\forall r = 0, \dots, m-1 : p_{k_r, k_{r+1}} > 0$$

By (*), $k_r \in C$ for any $r = 0, \dots, m$, and then $l \in C$.

$\therefore C$ is closed. □

For X a hom. M.C. we set

$$T_j = \min_{n \geq 1} \{X_n = j\} \text{ (for a fixed } j, \text{ the first time to reach } j\text{) (first passage time)}$$

$$f_{i,j}(n) := P(T_j = n | X_0 = i) \text{ (first passage probability)}$$

$$=: P_i(T_j = n)$$

Def. A state $i \in S$ is recurrent if $P_i(T_i < \infty) = 1$
and is transient otherwise.

Thm. The state i is recurrent iff $\sum_{n \in \mathbb{N}} P_{i,i}(n) = \infty$

Remark: We proved sth similar in Chapter 9 with 2 generating functions.

Pf. Let $i, j \in S$, we set

$$P_{i,j}(s) := \sum_{n \in \mathbb{N}} P_{i,j}(n) s^n \text{ with } P_{i,j}(0) = \delta_{i,j}$$

$$F_{i,j}(s) := \sum_{n \in \mathbb{N}} f_{i,j}(n) s^n \text{ with } f_{i,j}(0) = 0$$

Observe that

$$f_{i,j} := F_{i,j}(1) = \sum_{n \in \mathbb{N}} f_{i,j}(n) = \sum_{n \in \mathbb{N}} P_i(T_j = n) = P_i(T_j < \infty)$$

So i is recurrent iff $f_{i,i} = 1$.