

X: 1D random walk

Discrete time, discrete space process



For each $n \in \mathbb{N}$, a particle can either move to the left with probability q or to the right with probability p , ($p+q=1$), independently of the previous jumps or of the position.

Let S_n denote the position of the particle at time n . Then

$$S_{n+1} = \begin{cases} S_n + 1 & \text{with prob. } p \\ S_n - 1 & \text{with prob. } q \end{cases} \quad \text{and } S_n = S_0 + X_1 + X_2 + \dots + X_n$$

with X_1, \dots, X_n independent d.r.v., $\mathbb{P}(X_i=1) = p$, $\mathbb{P}(X_i=-1) = q = p-1$

The sequence of S_0, S_1, S_2, \dots is called a 1D simple random walk

which is symmetric if $p=q=\frac{1}{2}$

Thm. $v_n := \mathbb{P}(S_n = S_0)$, then ~~for~~ $\forall m \in \mathbb{N}$:

$$v_{2m+1} = 0, v_{2m} = \binom{2m}{m} p^m q^m$$

Proof. Wlog (\Leftrightarrow Without loss of generality) assume $S_0 = 0$

$v_{2m+1} = 0$ is clear;

$S_{2m} = \sum X_1 + X_2 + \dots + X_{2m}$ and then

$S_{2m} = 0$ iff m of $\sum X_j$ are equal to $+1$ and m of X_k are equal to -1

There are $\binom{2m}{m} = \frac{2m!}{m! m!}$ ways of choosing m among $2m$,

and the probability of each of these paths is $p^m q^m$.

$$\text{So } v_{2m} = \binom{2m}{m} p^m q^m.$$

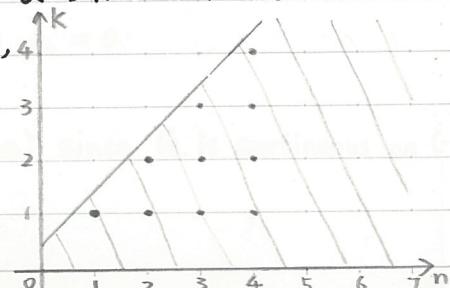
Other approach

Observe that $\frac{1}{2}(X_j + 1)$ is a Bernoulli distribution of parameter p

Since a sum of Bernoulli distribution is a binomial distribution

of parameters n and p , so if $S_0 = 0$,

$$B_n = \sum_{j=1}^n \frac{1}{2}(X_j + 1) = \frac{1}{2}(S_n + n)$$



Then

$$\begin{aligned} \mathbb{P}(BS_n = k | S_0 = 0) &= \mathbb{P}(B_n = \frac{k+n}{2} | S_0 = 0) \\ &= \binom{n}{\frac{1}{2}(n+k)} p^{\frac{1}{2}(n+k)} q^{n-\frac{1}{2}(n+k)} = \binom{n}{\frac{1}{2}(n+k)} p^{\frac{1}{2}(n+k)} q^{\frac{1}{2}(n-k)} \end{aligned}$$

which $\neq 0$ iff $\frac{1}{2}(n+k)$ is an integer between 0 and n .

Remark

A random walk is recurrent if it revisits its initial position with probability 1. Otherwise it is called transient.

Recurrent : $\Leftrightarrow P(\exists n \in \mathbb{N}^*: S_n = S_0) = 1 \Leftrightarrow$ Transient

Assume $S_0 = 0$, observe $E(X_j) = p - q$ and $\text{var}(X_j) < \infty$

$\Rightarrow \frac{1}{n} S_n \rightarrow p - q$ in mean square, by the law of large numbers.

Then, if $p-q > 0$, then the particle tends to go to $+\infty$; if $p-q < 0$, then the particle tends to go to $-\infty$;

If $p=q=\frac{1}{2}$, then $\frac{1}{n}S_n \rightarrow 0$ in mean square.

Thm. The probability that a 1D simple r.w. ever revisits initial position

$$\mathbb{P}(\exists n \in \mathbb{N}^*: S_n = 0 \mid S_0 = 0) = 1 - |p-q|$$

In particular, it is recurrent iff $p=q=\frac{1}{2}$.

Proof. Assume $S_0 = 0$, and set

$$A_n := \{S_n = 0\}, \quad U_n := P(A_n), \text{ see Thm 1}$$

$$B_n := \{S_n = 0 \wedge \forall k \in \{1, 2, \dots, n-1\} : S_k \neq 0\}, \quad f_n := P(B_n)$$

remark: the B_k are disjoint. ($\forall i, j \in \mathbb{N}^* : B_i \cap B_j = \emptyset$) so

$$\therefore P(A_n) = \sum_{k=1}^n P(A_n \cap B_k) = \sum_{k=1}^n P(B_k)P(A_{n-k})$$

$\therefore u_n = \sum_{k=1}^n f_k u_{n-k}$ for any $n = 1, 2, 3, \dots$. Now we look for f_k .

$$\therefore \sum_{n=1}^{\infty} u_n S^n = \sum_{n=1}^{\infty} \sum_{\substack{k=1 \\ k \\ n}}^n f_k u_{n-k} S^k S^{n-k} = \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} f_k S^k u_{n-k} S^{n-k}$$

$$\sum_{k=1}^{\infty} \sum_{l=0}^{\infty} f_k S^k u_l S^l = F(S) U(S) \Leftrightarrow F(S) = 1 - \frac{1}{U(S)}$$

$$\text{Now, } U(S) = (1 - 4pqS^2)^{-\frac{1}{2}} \Rightarrow F(S) = 1 - (1 - 4pqS^2)^{\frac{1}{2}}$$

Then

$$\begin{aligned} & P(\exists n \in \mathbb{N}^*: S_n = 0 | S_0 = 0) \\ &= P\left(\bigcup_{j=1}^{\infty} B_j\right) = \sum_{j=1}^{\infty} f_j = \lim_{S \rightarrow 1^-} \sum_{j=1}^{\infty} f_j S^j = \lim_{S \rightarrow 1^-} F(S) \end{aligned}$$

$$= 1 - \sqrt{1 - 4pq} = 1 - \sqrt{(p+q)^2 - 4pq} = 1 - \sqrt{(p-q)^2} = 1 - |p-q|$$

Remark: Consider $p = q = \frac{1}{2}$ and set $T := \min\{n \geq 1 | S_n = 0\}$

We have shown $P(T < \infty) = 1$

$$\text{Now } E(T) = \sum_{n=1}^{\infty} n f_n = \lim_{S \rightarrow 1^-} \sum_{n=1}^{\infty} n f_n S^{n-1} = \lim_{S \rightarrow 1^-} F'(S)$$

$$= \lim_{S \rightarrow 1^-} -\frac{1}{2} (1 - 4pqS^2)^{-\frac{1}{2}} \cdot 4pq \cdot 2S = \infty$$

Random walk with boundaries:

The gambler's ruin problem

2 players A and B, A has \$a and B has \$(N-a)

\Rightarrow Total capital is \$N > \$1

They play with a coin, it comes to heads with prob. p and tails with prob. q.

At each heads, B give 1\$ to A.

At each tails, A gives 1\$ to B.

The game ends when one has no more \$.

This corresponds to a random walk on $\{0, 1, 2, \dots, N\}$ starting at $a \in \{1, \dots, N-1\}$.

0 and N are called absorbing boundaries.

A wins if N is reached; B wins if 0 is reached.

We denote by $v(a)$ the probability that A wins with initial a.

Thm.

$$v(a) = \begin{cases} \frac{(\frac{q}{p})^a - 1}{(\frac{q}{p})^N - 1} & \text{if } p \neq q \\ \frac{a}{N} & \text{if } p = q = \frac{1}{2} \end{cases}$$

What happens if $N = \infty$, then only B can win because only 1 absorbing barrier.
Thm (when $N = \infty$)

The prob. $\Pi(a)$ that A loses ($a=0$) is given by

$$\Pi(a) = \begin{cases} (\frac{q}{p})^a & \text{if } p > q \\ 1 & \text{if } p \leq q \end{cases} \quad \text{Formally } \Pi(a) = \lim_{N \rightarrow \infty} 1 - v(a) \quad \square$$

Proof of the first thm:

Let H be the event that the joint flip is a heads

We assume that $a \neq 0, a \neq N$

$$v(a) = \underbrace{\mathbb{P}(A \text{ wins} | H)}_{v(a+1)} \underbrace{\mathbb{P}(H)}_p + \underbrace{\mathbb{P}(A \text{ wins} | H^c)}_{v(a-1)} \underbrace{\mathbb{P}(H^c)}_q$$

$$\Leftrightarrow v(a+1) - \frac{1}{p} v(a) + \frac{q}{p} v(a-1) = 0 \quad \forall a \in \{1, \dots, N-1\}$$

$$\Leftrightarrow v(a+2) - \frac{1}{p} v(a+1) + \frac{q}{p} v(a) = 0 \quad \forall a \in \{0, \dots, N-2\} \quad (*)$$

This is called a difference equation.

Solve the equation $x^2 - \frac{1}{p}x + \frac{q}{p} = 0 \Leftrightarrow x = 1 \text{ or } \frac{q}{p}$ (note $p+q=1$)

So general solution of (*) is

$$v(a) = \begin{cases} \alpha 1^a + \beta (\frac{q}{p})^a & \text{if } p \neq q \\ \alpha + \beta a & \text{if } p = q \end{cases} \quad (\alpha, \beta \in \mathbb{C})$$

Boundary condition: $v(0) = 0, v(N) = 1$

Then we can deduce $v(a)$. □