Nagoya University, G30 program Fall 2017

Calculus I Instructor : Serge Richard

Final examination

Exercise 1 *Compute the derivative of the following functions (and simplify the results, if possible):*

$$a)\ x \mapsto \cos\big((x+1)^2\big), \qquad b)\ x \mapsto \frac{\mathrm{e}^{-x}+1}{\mathrm{e}^{x}+1}, \qquad c)\ x \mapsto x^{2x}.$$

Exercise 2 *Compute the following limits (and justify your computation):*

$$a)\ \lim_{x\to 0} \frac{\ln(1+x)}{x^2-2x}, \qquad\qquad b)\ \lim_{x\to 0} \frac{\cos(x^2)-1}{x^4}.$$

Exercise 3 *Compute the following integrals:*

$$a)\ \int_0^{\sqrt{\pi}} x\sin(2x^2)\mathrm{d}x, \qquad b)\ \int x\ln(x^\alpha)\mathrm{d}x \quad \text{for any } \alpha \geq 0, \qquad c)\ \int_0^{\pi} \sin^3(x)\,\mathrm{d}x.$$

Exercise 4 *Consider the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x - x^2$ and $g(x) = x$.*

(i) Sketch (approximately) these functions.

(ii) Determine the intersections of the 2 curves, which means the solutions of $f(x) = g(x)$.

(iii) Compute the area between f and g, which means the area of the surface below the curve defined by f but above the curve defined by g.

Exercise 5 *Consider the power series $\sum_{n=0}^{\infty} \frac{(-3)^n}{\sqrt{n+1}}\, x^n$. Answer the following questions with justification.*

(i) What is the radius of convergence of this power series ?

(ii) If $r > 0$ denotes the radius of convergence found in (i), check if the series $\sum_{n=0}^{\infty} \frac{(-3)^n}{\sqrt{n+1}}\, r^n$ is convergent and check if $\sum_{n=0}^{\infty} \frac{(-3)^n}{\sqrt{n+1}}\, (-r)^n$ is convergent ?

Exercise 6 (Each part of this exercise is independent from the previous ones) *The aim of this exercise is to study the function $f : [0,\infty) \to \mathbb{R}$ given by $f(x) := \mathrm{e}^{-1/x} = \mathrm{e}^{-x^{-1}}$ for any $x > 0$.*

(i) Show that $\lim_{x\to 0_+} f(x) = 0$. We thus set $f(0) := 0$.

(ii) Compute $\lim_{x\to\infty} f(x)$ and also f', and sketch (approximately) the function f.

(iii) Show that for any $n \in \mathbb{N}$ one has $\lim_{x\to 0_+} \mathrm{e}^{-1/x}\, x^{-n} = 0$.

(iv) Based on the result of (iii) compute the Taylor polynomial of f at 0 for any $x > 0$, namely $P_n(0,x)$, for any $n \in \mathbb{N}$. What do you observe ?

Total 26 pts Final

Exercise 1 6 pts

2 a) $\cos\left((x+1)^2\right)' = -\sin\left((x+1)^2\right) 2(x+1) = -2(x+1)\sin\left((x+1)^2\right)$.

2 b) $\left(\frac{e^{-x}+1}{e^x+1}\right)' = \frac{-e^{-x}(e^x+1) - e^x(e^{-x}+1)}{(e^x+1)^2}$

$= \frac{-1 - e^{-x} - 1 - e^x}{(e^x+1)^2} = -\frac{e^x + e^{-x} + 2}{(e^x+1)^2}$.

2 c) $(x^{2x})' = \left(e^{\ln(x^{2x})}\right)' = \left(e^{2x\ln(x)}\right)' = e^{2x\ln(x)}(2\ln(x)+2)$

$= 2x^{2x}(\ln(x)+1)$.

Exercise 2 3 pts

a) $\lim_{x\to 0} \frac{\ln(1+x)}{x^2-2x} \overset{\text{Hopital's rule}}{=} \lim_{x\to 0} \frac{\frac{1}{1+x}}{2x-2} = -\frac{1}{2}$. 1

b) $\lim_{x\to 0} \frac{\cos(x^2)-1}{x^4} = \lim_{x\to 0} \frac{1 - \frac{(x^2)^2}{2} + R_4(0,x^2) - 1}{x^4} = -\frac{1}{2}$. 1

↑ Taylor expansion

1 for argumentation

Exercise 3 5 pts

a) $\int_0^{\sqrt{\pi}} x\sin(2x^2)\,dx = -\frac{1}{4}\cos(2x^2)\Big|_0^{\sqrt{\pi}} = 0$. 1

b) $\int x\ln(x^\alpha)\,dx = \frac{1}{2}x^2\ln(x^\alpha) - \int \frac{1}{2}x^2 \frac{1}{x^\alpha}\alpha x^{\alpha-1}$

$= \frac{1}{2}x^2\ln(x^\alpha) - \frac{\alpha}{2}\int x^{2-\alpha+\alpha-1}\,dx$

$= \frac{1}{2}x^2\ln(x^\alpha) - \frac{\alpha}{4}x^2 + \text{cst}$. 2

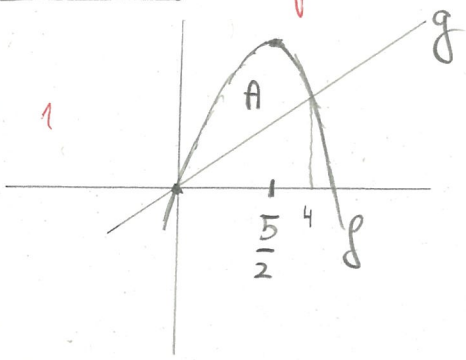
Or use directly $\int x\ln(x^\alpha)\,dx = \alpha\int x\ln(x)\,dx$.

c) $\int_0^\pi \sin^3(x)\,dx = \int_0^\pi \sin(x)(1-\cos^2(x))\,dx = -\cos(x)\Big|_0^\pi + \frac{1}{3}\cos^3(x)\Big|_0^\pi$

$= -(-1-1) + \frac{1}{3}(-1-1) = 2 - \frac{2}{3} = \frac{4}{3}$. 2

Exercise 4 — 4 pts

i) 1



1 ii) $f(x) = g(x)$

$\Leftrightarrow 5x - x^2 = x$

$\Leftrightarrow 4x - x^2 = 0$

$\Leftrightarrow x(4-x) = 0$

The solutions are 0 and 4, and the intersections are the points $(0,0)$ and $(4,4)$ in $\mathbb{R}^2$.

iii) 2

$$A = \int_0^4 f(x)\,dx - \int_0^4 g(x)\,dx$$

$$= \int_0^4 (5x - x^2 - x)\,dx$$

$$= \int_0^4 (4x - x^2)\,dx$$

$$= \left(2x^2 - \tfrac{1}{3}x^3\right)\Big|_0^4 = 32 - \frac{64}{3} = \frac{96-64}{3} = \frac{32}{3}.$$

Exercise 5 — 4 pts

i) By the ratio test: $\dfrac{|(-3)|^{n+1}}{\sqrt{n+2}} \dfrac{\sqrt{n+1}}{|(-3)|^n} r = 3 \dfrac{\sqrt{n+1}}{\sqrt{n+2}} r$.

Since $\dfrac{\sqrt{n+1}}{\sqrt{n+2}} = \dfrac{\sqrt{1+1/n}}{\sqrt{1+2/n}} \xrightarrow{n\to\infty} 1$, the radius of convergence is $\frac{1}{3}$. 1

ii) For $r = \frac{1}{3}$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{\sqrt{n+1}} \left(\tfrac{1}{3}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ which converges 1

since it is an alternating series (and the 3 conditions are satisfied.

On the other hand, $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{\sqrt{n+1}} \left(-\tfrac{1}{3}\right)^n = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \infty$

by the criterion of the integral. 1

The series converges for $x \in (-\frac{1}{3}, \frac{1}{3}]$.

1 for argument

Exercise 6 — 4 pts

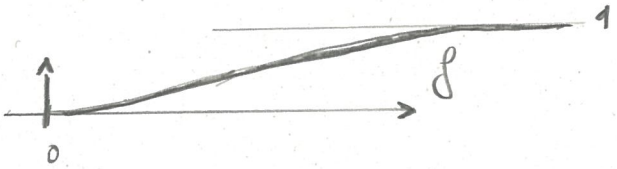
i) $\lim_{x \to 0_+} f(x) = \lim_{x \to 0_+} e^{-x^{-1}} = \lim_{y \to \infty} e^{-y} = \lim_{y \to \infty} \frac{1}{e^y} = 0.$ (1)

set $y = \frac{1}{x}$

ii) $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{-1/x} = e^0 = 1$ (1)

Observe that $f'(x) = e^{-x^{-1}} \left(\frac{1}{x^2}\right) = \frac{1}{x^2} e^{-1/x} > 0$

$\Rightarrow$ the function is increasing



iii) $\lim_{x \to 0_+} e^{-1/x} x^{-n} = \lim_{y \to \infty} e^{-y} y^n = \lim_{y \to \infty} \frac{y^n}{e^y} = 0$ (1)

the exponential grows faster than any polynomial.

iv) One observes easily that for any $j \in \mathbb{N}^*$

$f^{(j)}(x) = \sum_{k=1}^{n} e^{-1/x} x^{-k}$ for some n depending on j. (1)

By (iii), $\lim_{x \to 0_+} f^{(j)}(x) = 0$, and thus all coefficients in the Taylor polynomial are 0.

$\Rightarrow P_n(0, x) = 0$ for any n.

This function f has no Taylor expansion.