CHAPTER VI

Sketching Curves

We have developed enough techniques to be able to sketch curves and graphs of functions much more efficiently than before. We shall investigate systematically the behavior of a curve, and the mean value theorem will play a fundamental role.

We shall especially look for the following aspects of the curve:

1. Intersections with the coordinate axes.
2. Critical points.
3. Regions of increase.
4. Regions of decrease.
5. Maxima and minima (including the local ones).
6. Behavior as x becomes large positive and large negative.
7. Values of x near which y becomes large positive or large negative.

These seven pieces of information will be quite sufficient to give us a fairly accurate idea of what the graph looks like. We shall devote a section to considering one other aspect, namely:

8. Regions where the curve is bending up or down.

VI, §1. BEHAVIOR AS x BECOMES VERY LARGE

Suppose we have a function f defined for all sufficiently large numbers. Then we get substantial information concerning our function by investigating how it behaves as x becomes large.

For instance, $\sin x$ oscillates between -1 and $+1$ no matter how large x is.

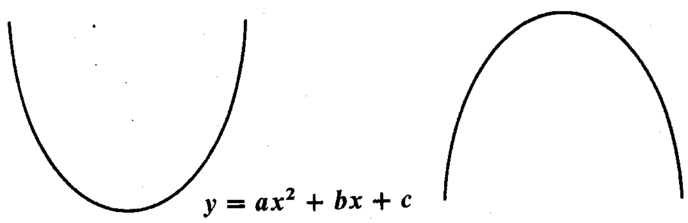
However, polynomials don't oscillate. When $f(x) = x^2$, as x becomes large positive, so does x^2. Similarly with the function x^3, or x^4 (etc.). We consider this systematically.

Parabolas

Example 1. Consider a parabola,

$$y = ax^2 + bx + c,$$

with $a \neq 0$. There are two essential cases, when $a > 0$ and $a < 0$. We shall see that the parabola looks like those drawn in the figure.



Case $a > 0$ Case $a < 0$

We look at numerical examples.

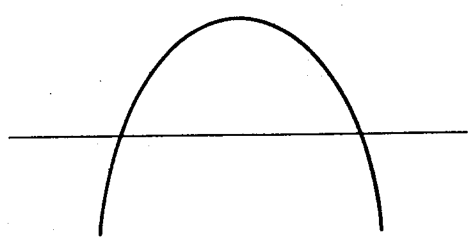
Example 2. Sketch the graph of the curve

$$y = f(x) = -3x^2 + 5x - 1.$$

We recognize this as a parabola. Factoring out x^2 shows that

$$f(x) = x^2\left(-3 + \frac{5}{x} - \frac{1}{x^2}\right).$$

When x is large positive or negative, then x^2 is large positive and the factor on the right is close to -3. Hence $f(x)$ is large negative. This means that the parabola has the shape as shown on the figure.



We have $f'(x) = -6x + 5$. Thus $f'(x) = 0$ if and only if $-6x + 5 = 0$, or in other words,

$$x = \tfrac{5}{6}.$$

There is exactly one critical point. We have

$$f(\tfrac{5}{6}) = -3(\tfrac{5}{6})^2 + \tfrac{25}{6} - 1 > 0.$$

The critical point is a maximum, because we have already seen that the parabola bends down.

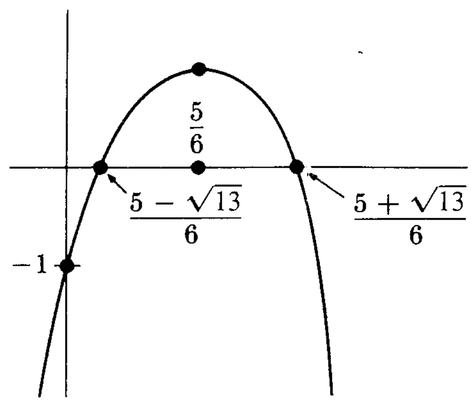
The curve crosses the x-axis exactly when

$$-3x^2 + 5x - 1 = 0.$$

By the quadratic formula (see Chapter II, §8), this is the case when

$$x = \frac{-5 \pm \sqrt{25 - 12}}{-6} = \frac{5 \pm \sqrt{13}}{6}.$$

Hence the graph of the parabola looks as on the figure.



The same principle applies to sketching any parabola.

(i) Looking at what happens when x becomes large positive or negative tells us whether the parabola bends up or down.

(ii) A quadratic function

$$f(x) = ax^2 + bx + c, \qquad \text{with} \quad a \neq 0$$

has only one critical point, when

$$f'(x) = 2ax + b = 0$$

so when

$$x = -b/2a.$$

Knowing whether the parabola bends up or down tells us whether the critical point is a maximum or minimum, and the value $x = -b/2a$ tells us exactly where this critical point lies.

(iii) The points where the parabola crosses the x-axis are determined by the quadratic formula.

Example 3. Cubics. Consider a polynomial

$$f(x) = x^3 + 2x - 1.$$

We can write it in the form

$$x^3\left(1 + \frac{2}{x^2} - \frac{1}{x^3}\right).$$

When x becomes very large, the expression

$$1 + \frac{2}{x^2} - \frac{1}{x^3}$$

approaches 1. In particular, given a small number $\delta > 0$, we have, for all x sufficiently large, the inequality

$$1 - \delta < 1 + \frac{2}{x^2} - \frac{1}{x^3} < 1 + \delta.$$

Therefore $f(x)$ satisfies the inequality

$$x^3(1 - \delta) < f(x) < x^3(1 + \delta).$$

This tells us that $f(x)$ behaves very much like x^3 when x is very large. In particular:

If x becomes large positive, then $f(x)$ becomes large positive.

If x becomes large negative, then $f(x)$ becomes large negative.

A similar argument can be applied to any polynomial.

It is convenient to use an abbreviation for the expression "become large positive." Instead of say x becomes large positive, we write

$$x \to \infty$$

and also say that x **approaches,** or **goes to infinity**. **Warning: there is no number called infinity**. The above symbols merely abbreviate the notion of becoming large positive. We have a similar notation for x becoming large negative, when we write

$$x \to -\infty$$

and say that x **approaches minus infinity**. Thus in the case when

$$f(x) = x^3 + 2x - 1,$$

we can assert:

$$\text{If } x \to \infty \quad \text{then } f(x) \to \infty.$$
$$\text{If } x \to -\infty \text{ then } f(x) \to -\infty.$$

Example 4. Consider a quotient of polynomials like

$$Q(x) = \frac{x^3 + 2x - 1}{2x^3 - x + 1}.$$

We factor out the highest power of x from the numerator and denominator, and therefore write $Q(x)$ in the form

$$Q(x) = \frac{x^3(1 + 2/x^2 - 1/x^3)}{x^3(2 - 1/x^2 + 1/x^3)} = \frac{1 + 2/x^2 - 1/x^3}{2 - 1/x^2 + 1/x^3}.$$

As x becomes very large, the numerator approaches 1 and the denominator approaches 2. Thus our fraction approaches $\frac{1}{2}$. We may express this in the form

$$\lim_{x \to \infty} Q(x) = \frac{1}{2}.$$

Or we may write:

$$\text{If } x \to \pm\infty \text{ then } Q(x) \to \tfrac{1}{2}.$$

Example 5. Consider the quotient

$$Q(x) = \frac{x^2 - 1}{x^3 - 2x + 1}.$$

Does it approach a limit as x becomes very large?

We write

$$Q(x) = \frac{x^2(1 - 1/x^2)}{x^3(1 - 2/x^2 + 1/x^3)}$$
$$= \frac{1}{x} \frac{1 - 1/x^2}{1 - 2/x^2 + 1/x^3}.$$

As x becomes large, the term $1/x$ approaches 0, and the other factor approaches 1. Hence $Q(x)$ approaches 0 as x becomes large negative or positive.

We may also write

$$\text{If } x \to \pm\infty \text{ then } Q(x) \to 0,$$

or

$$\lim_{x \to \pm\infty} Q(x) = 0.$$

Example 6. Consider the quotient

$$Q(x) = \frac{x^3 - 1}{x^2 + 5}$$

and determine what happens when x becomes large.

We write

$$\begin{aligned} Q(x) &= \frac{x^3(1 - 1/x^3)}{x^2(1 + 5/x^2)} \\ &= x\,\frac{1 - 1/x^3}{1 + 5/x^2}. \end{aligned}$$

As x becomes large, positive or negative, the quotient

$$\frac{1 - 1/x^3}{1 + 5/x^2}$$

approaches 1. Hence $Q(x)$ differs from x by a factor near 1. Hence $Q(x)$ becomes large positive when x is large positive, and becomes large negative when x is large negative. We may express this by saying:

$$\text{If } x \to \infty \quad \text{then } Q(x) \to \infty.$$

$$\text{If } x \to -\infty \text{ then } Q(x) \to -\infty.$$

We may also write these assertions in the form of a limit:

$$\lim_{x \to \infty} Q(x) = \infty \qquad \text{and} \qquad \lim_{x \to -\infty} Q(x) = -\infty.$$

However, even though we use this notation, and may say that the limit of $Q(x)$ is $-\infty$ when x becomes large negative, we emphasize that $-\infty$ is not a number, and so this limit is not quite the same as when the limit is a number. It is correct to say that there is no number which is the limit of $Q(x)$ as $x \to \infty$ or as $x \to -\infty$.

These four examples are typical of what happens when we deal with quotients of polynomials.

Later when we deal with exponents and logarithms, we shall again meet the problem of comparing the quotient of two expressions which become large. There will be a common ground for some of the arguments, summarized by the following table:

Large positive times large positive is large positive. Large positive times large negative is large negative. Large negative times large negative is large positive. Small positive times large positive: you can't tell without knowing more information.

VI, §1. EXERCISES

Find the limits of the following quotients $Q(x)$ as x becomes large positive or negative. In other words, find

$$\lim_{x \to \infty} Q(x) \quad \text{and} \quad \lim_{x \to -\infty} Q(x).$$

1. $\dfrac{2x^3 - x}{x^4 - 1}$
2. $\dfrac{\sin x}{x}$
3. $\dfrac{\cos x}{x}$
4. $\dfrac{x^2 + 1}{\pi x^2 - 1}$
5. $\dfrac{\sin 4x}{x^3}$
6. $\dfrac{5x^4 - x^3 + 3x + 2}{x^3 - 1}$
7. $\dfrac{-x^2 + 1}{x + 5}$
8. $\dfrac{2x^4 - 1}{-4x^4 + x^2}$
9. $\dfrac{2x^4 - 1}{-4x^3 + x^2}$
10. $\dfrac{2x^4 - 1}{-4x^5 + x^2}$

Describe the behavior of the following polynomials as x becomes large positive and large negative.

11. $x^3 - x + 1$
12. $-x^3 - x + 1$
13. $x^4 + 3x^3 + 2$
14. $-x^4 + 3x^3 + 2$
15. $2x^5 + x^2 - 100$
16. $-3x^5 + x + 1000$
17. $10x^6 - x^4$
18. $-3x^6 + x^3 + 1$

19. A function $f(x)$ which can be expressed as follows:

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0,$$

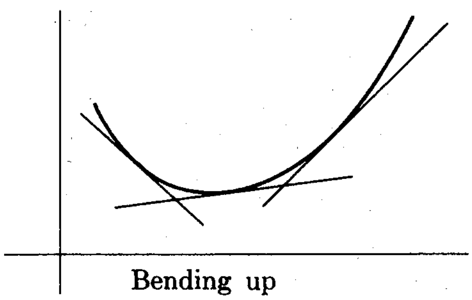
where n is a positive integer and the $a_n, a_{n-1}, \ldots, a_0$ are numbers, is called a polynomial. If $a_n \neq 0$, then n is called the **degree** of the polynomial. Describe the behavior of $f(x)$ as x becomes large positive or negative, n is odd or even, and $a_n > 0$ or $a_n < 0$. You will have eight cases to consider. Fill out the following table.

n	a_n	$x \to \infty$	$x \to -\infty$
Odd	> 0	$f(x) \to ?$	$f(x) \to ?$
Odd	< 0	$f(x) \to ?$	$f(x) \to ?$
Even	> 0	$f(x) \to ?$	$f(x) \to ?$
Even	< 0	$f(x) \to ?$	$f(x) \to ?$

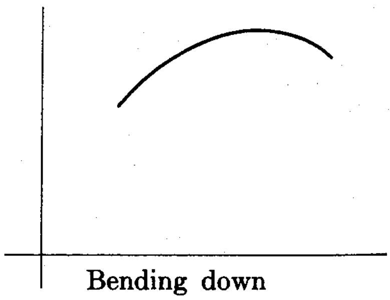
20. Using the intermediate value theorem, show that any polynomial of odd degree has a root.

VI, §2. BENDING UP AND DOWN

Let a, b be numbers, $a < b$. Let f be a continuous function defined on the interval $[a, b]$. Assume that f' and f'' exist on the interval $a < x < b$. We view the second derivative f'' as the rate of change of the slope of the curve $y = f(x)$ over the interval. If the second derivative is positive in the interval $a < x < b$, then the slope of the curve is increasing, and we interpret this as meaning that the curve is **bending up**. If the second derivative is negative, we interpret this as meaning that the curve is **bending down**. The following two figures illustrate this.



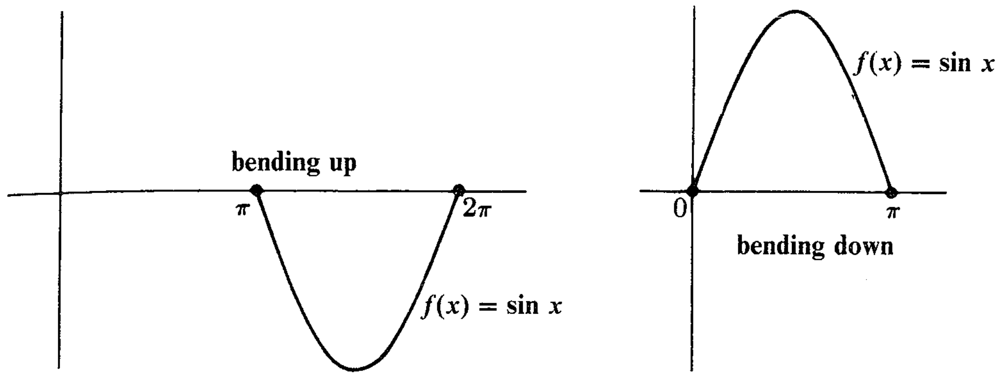
Bending up



Bending down

Example 1. The curve $y = x^2$ is bending up. We can see this using the second derivative. Let $f(x) = x^2$. Then $f''(x) = 2$, and the second derivative is always positive. The present considerations justify drawing the curve as we have always done, i.e. bending up.

Example 2. Let $f(x) = \sin x$. We have $f''(x) = -\sin x$, and thus $f''(x) > 0$ on the interval $\pi < x < 2\pi$. Hence the curve is bending up on this interval. Similarly, $f''(x) < 0$ on the interval $0 < x < \pi$. Hence the curve is bending down on this interval, as shown on the next figures. Of course, this merely justifies the drawings which we have always made for the graph of the sine function.



Example 3. Determine the intervals where the curve

$$y = -x^3 + 3x - 5$$

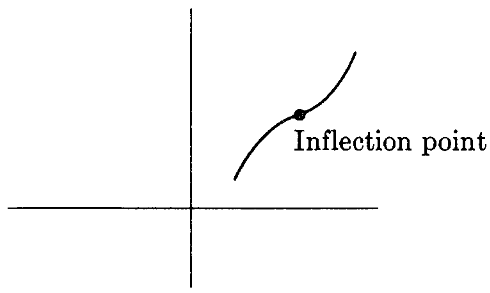
is bending up and bending down.

Let $f(x) = -x^3 + 3x - 5$. Then $f''(x) = -6x$. Thus:

$$f''(x) > 0 \quad \Leftrightarrow \quad x < 0,$$
$$f''(x) < 0 \quad \Leftrightarrow \quad x > 0.$$

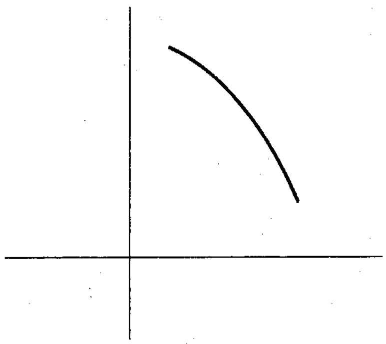
Hence f is bending up if and only if $x < 0$; and f is bending down if and only if $x > 0$. The graph of this curve will be discussed fully in the next section when we graph cubics systematically.

A point where a curve changes its behavior from bending up to down (or vice versa) is called an **inflection point**. If the curve is the graph of a function f whose second derivative exists and is continuous, then we must have $f''(x) = 0$ at that point. The following picture illustrates this:

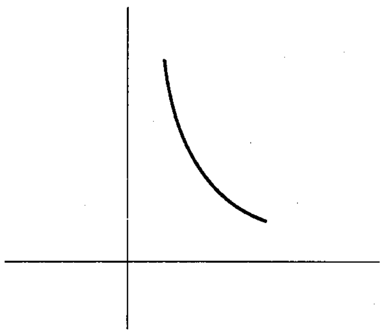


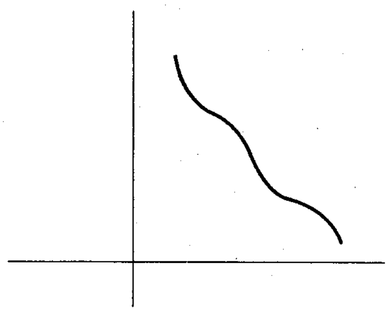
In Example 3 above, the point $(0, -5)$ is an inflection point.

The determination of regions of bending up or down and inflection points gives us worthwhile pieces of information concerning curves. For instance, knowing that a curve in a region of decrease is actually bending down tells us that the decrease occurs essentially as in this example:



and not as in these examples:





The second derivative can also be used as a test whether a critical point is a **local** maximum or minimum.

Second derivative test. *Let f be twice continuously differentiable on an open interval, and suppose that c is a point where*

$$f'(c) = 0 \qquad \textit{and} \qquad f''(c) > 0.$$

Then c is a local minimum point of f. On the other hand, if

$$f''(c) < 0$$

then c is a local maximum point of f.

To see this, suppose that $f''(c) > 0$. Then $f''(x) > 0$ for all x close to c because we assumed that the second derivative is continuous. Thus the curve is bending up. Consequently the picture of the graph of f is as on Fig. 1(a) and c is a local minimum.

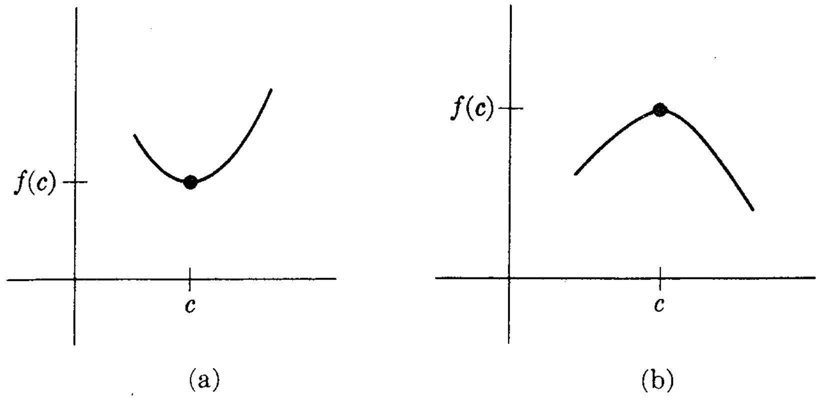


Figure 1

A similar argument shows that if $f''(c) < 0$ then c is a local maximum as on Fig. 1(b).

VI, §2. EXERCISES

1. Determine all inflection points of $\sin x$.
2. Determine all inflection points of $\cos x$.
3. Determine the inflection points of $f(x) = \tan x$ for $-\pi/2 < x < \pi/2$.
4. Sketch the curve $y = \sin^2 x$. Determine the critical points and the inflection points. Compare with the graph of $|\sin x|$.
5. Sketch the curve $y = \cos^2 x$. Determine the critical points and the inflection points. Compare with the graph of $|\cos x|$.

Determine the inflection points and the intervals of bending up and bending down for the following curves.

6. $y = x + \dfrac{1}{x}$
7. $y = \dfrac{x}{x^2+1}$
8. $y = \dfrac{x}{x^2-1}$

9. Sketch the curve $y = f(x) = \sin x + \cos x$ for $0 \leqq x \leqq 2\pi$. First plot all values $f(n\pi/4)$ with $n = 0, 1, 2, 3, 4, 5, 6, 7, 8$. Then determine all the critical points. Then determine the regions of increase and decrease. Then determine the inflection points, and the regions where the curve bends up or down.

VI, §3. CUBIC POLYNOMIALS

We can now sketch the graphs of cubic polynomials systematically.

Example 1. Sketch the graph of $f(x) = x^3 - 2x + 1$.

1.

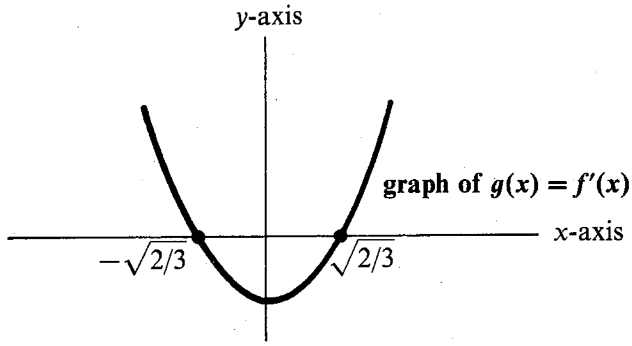
$$\text{If } x \to \infty \quad \text{then } f(x) \to \infty \quad \text{by §1.}$$
$$\text{If } x \to -\infty \text{ then } f(x) \to -\infty \text{ by §1.}$$

2. We have $f'(x) = 3x^2 - 2$. Thus

$$f'(x) = 0 \quad \Leftrightarrow \quad x = \pm\sqrt{2/3}.$$

The critical points of f are $x = \sqrt{2/3}$ and $x = -\sqrt{2/3}$.

3. Let $g(x) = f'(x) = 3x^2 - 2$. Then the graph of g is a parabola, and the x-intercepts of the graph of g are precisely the critical points of f. (Do not confuse the functions f and $f' = g$.) The graph of g is a parabola bending up, as follows.



Therefore:

$$f'(x) > 0 \quad \Leftrightarrow \quad x > \sqrt{2/3} \quad \text{and} \quad x < -\sqrt{2/3}, \quad \text{where} \quad g(x) > 0$$

and f is strictly increasing on the intervals

$$x \geqq \sqrt{2/3} \qquad \text{and} \qquad x \leqq -\sqrt{2/3}.$$

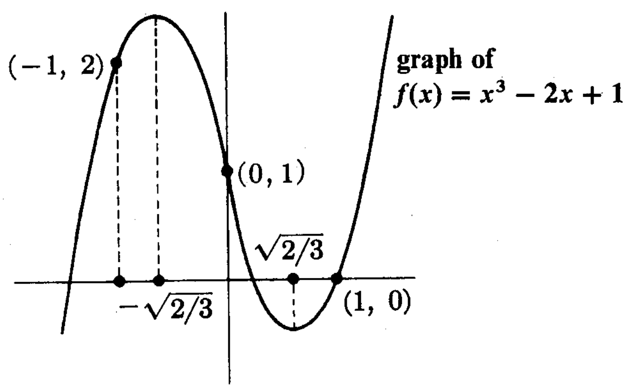
Similarly:

$$f'(x) < 0 \quad \Leftrightarrow \quad -\sqrt{2/3} < x < \sqrt{2/3}, \quad \text{where} \quad g(x) < 0,$$

and f is strictly decreasing on this interval. Therefore $-\sqrt{2/3}$ is a local maximum for f, and $\sqrt{2/3}$ is a local minimum.

4. $f''(x) = 6x$, and $f''(x) > 0$ if and only if $x > 0$. Also $f''(x) < 0$ if and only if $x < 0$. Therefore f is bending up for $x > 0$ and bending down for $x < 0$. There is an inflection point at $x = 0$.

Putting all this together, we find that the graph of f looks like this.



Observe how we used a quadratic polynomial, namely $f'(x) = 3x^2 - 2$, as an intermediate step in the arguments.

Remark 1. Instead of using the quadratic polynomial, we can also argue as follows, after we know that the only critical points of f are $x = \sqrt{2/3}$ and $x = -\sqrt{2/3}$. Consider the interval $x < -\sqrt{2/3}$. Then $f'(x) \neq 0$ for all $x < \sqrt{-2/3}$. Hence $f'(x)$ is either > 0 for all $x < -\sqrt{2/3}$, or $f'(x) < 0$ for all $x < -\sqrt{2/3}$ by the intermediate value theorem. Which is it? We just try one value, say with $x = -10$, to see that $f'(x) > 0$ for $x < -\sqrt{2/3}$, because $f'(-10) = 3 \cdot 10^2 - 2 = 298$. Hence we must have $f'(x) > 0$ for $x < -\sqrt{2/3}$.

Remark 2. For a cubic polynomial it is much more difficult to determine the roots, that is the x-intercepts, and we usually do not do so, unless there is a simple way of doing it, by accident. In the above case when

$$f(x) = x^3 - 2x + 1,$$

there is such an accident, since $f(1) = 0$. Therefore 1 is a root of f. Hence $f(x)$ factors

$$x^3 - 2x + 1 = (x - 1)(x^2 + x - 1).$$

The other roots of f are the roots of $x^2 + x - 1$, which can be found by the quadratic formula:

$$x = \frac{-1 \pm \sqrt{5}}{2}.$$

In the next example, however, there is no such simple way of finding the roots, and we do not find them.

Example 2. Sketch the graph of the curve

$$y = -x^3 + 3x - 5.$$

1. When $x = 0$, we have $y = -5$. With polynomials of degree $\geqq 3$ there is in general no simple formula for those x such that $f(x) = 0$, so we do not give explicitly the intersection of the graph with the x-axis.

2. The derivative is

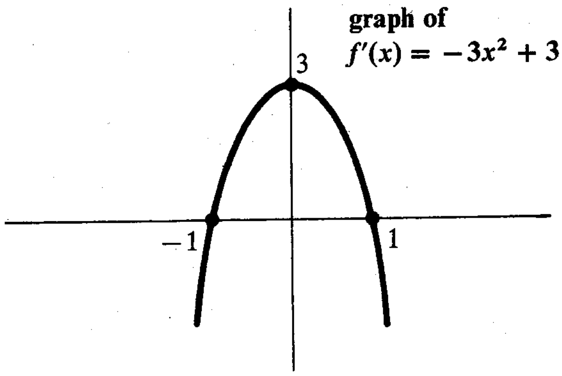
$$f'(x) = -3x^2 + 3.$$

The graph of $f'(x)$ is a parabola bending down, as you should know from previous experience with parabolas. We have

$$f'(x) = 0 \iff x^2 = 1 \iff x = 1 \text{ and } x = -1.$$

Thus there are two critical points of f, namely $x = 1$ and $x = -1$.

3. The graph of $f'(x)$ looks like a parabola bending down, as follows.



Then:

$$\begin{aligned} f \text{ is strictly decreasing} &\iff f'(x) < 0 \\ &\iff x < -1 \quad \text{and} \quad x > 1. \\ f \text{ is strictly increasing} &\iff f'(x) > 0 \\ &\iff -1 < x < 1. \end{aligned}$$

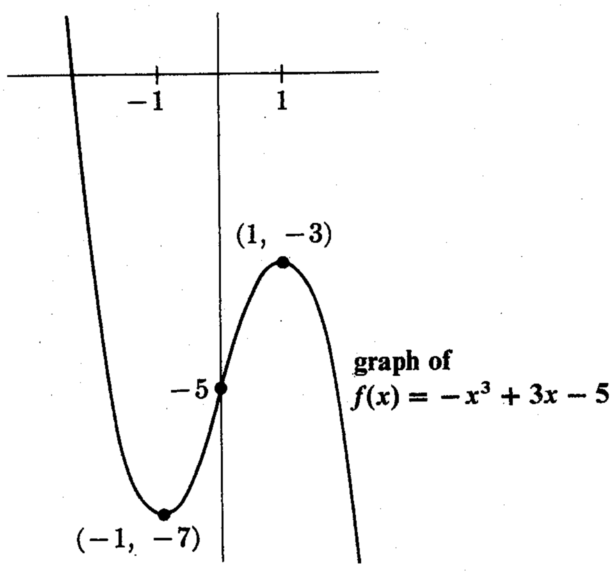
Therefore f has a local minimum at $x = -1$, and has a local maximum at $x = 1$.

4.

$$\begin{aligned} &\text{If } x \to \infty \quad \text{then } f(x) \to -\infty \text{ by §1.} \\ &\text{If } x \to -\infty \text{ then } f(x) \to +\infty \text{ by §1.} \end{aligned}$$

5. We have $f''(x) = -6x$. Hence $f''(x) > 0$ if and only if $x < 0$ and $f''(x) < 0$ if and only if $x > 0$. There is an inflection point at $x = 0$.

Putting all this information together, we see that the graph of f looks like this:



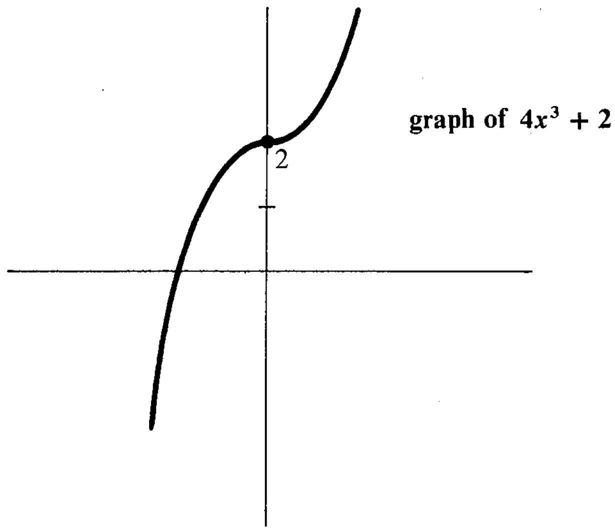
Remark. When f is a polynomial of degree 3, its derivative $f'(x)$ is a polynomial of degree 2, and in general this polynomial has two roots, giving the two critical points of the curve $y = f(x)$. In the preceding example, these critical points are at $(-1, -7)$ and $(1, -3)$.

Again note how we used the graph of a parabola, namely the graph of $f'(x)$, in the process of determining the graph of f itself.

In the last two examples, the cubic polynomial had two bumps, at the two critical points. This is the most general form of cubic polynomials. However, there may be special cases, when there is no critical point, or only one critical point.

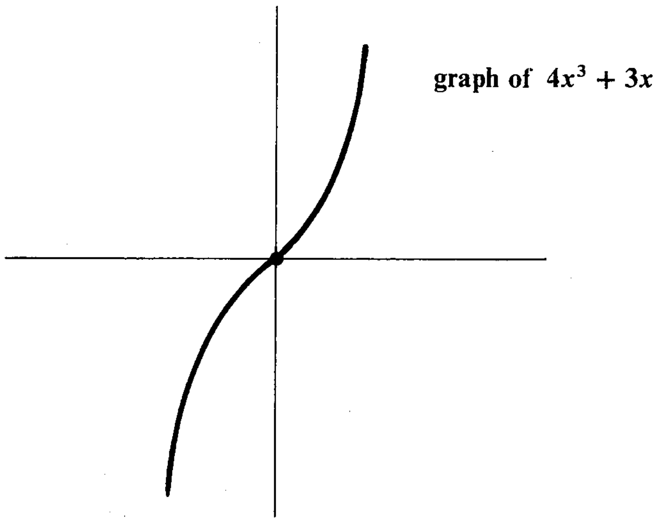
Example 3(a). Let $f(x) = 4x^3 + 2$. Sketch the graph of f.

Here we have $f'(x) = 12x^2 > 0$ for all $x \neq 0$. There is only one critical point, when $x = 0$. Hence the function is strictly increasing for all x, and its graph looks like this.



Example 3(b). Let $f(x) = 4x^3 + 3x$. Sketch the graph of f.

Here we have $f'(x) = 12x^2 + 3 > 0$ for *all* x. Therefore the graph of f looks like this. There is no critical point.



In both examples, we have

$$f''(x) = 24x.$$

Thus in both examples, there is an inflection point at $x = 0$. The graph of f bends down for $x < 0$ and bends up for $x > 0$. The difference between case (a) and case (b) is that in case (a) the inflection point is a critical point, where the derivative of f is equal to 0, so that the curve is flat at the critical point. In case (b), the derivative at the inflection point is

$$f'(0) = 3,$$

so in case (b) the derivative at the inflection point is positive.

VI, §3. EXERCISES

1. Show that a curve

$$y = ax^3 + bx^2 + cx + d$$

with $a \neq 0$ has exactly one inflection point.

Sketch the graphs of the following curves.

2. $x^3 - 2x^2 + 3x$
3. $x^3 + x^2 - 3x$
4. $2x^3 - x^2 - 3x$
5. $\frac{1}{3}x^3 + x^2 - 2x$
6. $x^3 - 3x^2 + 6x - 3$
7. $x^3 + x - 1$
8. $x^3 - x - 1$
9. $-x^3 + 2x + 5$
10. $-2x^3 + x + 2$
11. $x^3 - x^2 + 1$
12. $y = x^4 + 4x$
13. $y = x^5 + x$
14. $y = x^6 + 6x$
15. $y = x^7 + x$
16. $y = x^8 + x$
17. Which of the following polynomials have a minimum (for all x)?
(a) $x^6 - x + 2$ (b) $x^5 - x + 2$
(c) $-x^6 - x + 2$ (d) $-x^5 - x + 2$
(e) $x^6 + x + 2$ (f) $x^5 + x + 2$
Sketch the graphs of these polynomials.
18. Which of the polynomials in Exercise 17 have a maximum (for all x)?

In the following two problems:
(a) Show that f has exactly two inflection points.
(b) Sketch the graph of f. Determine the critical points explictly. Determine the regions of bending up or down.

19. $f(x) = x^4 + 3x^3 - x^2 + 5$
20. $f(x) = x^4 - 2x^3 + x^2 + 3$
21. Sketch the graph of the function

$$f(x) = x^6 - \tfrac{3}{2}x^4 + \tfrac{9}{16}x^2 - \tfrac{1}{32}.$$

Find the critical points. Find the values of f at these critical points. Sketch the graph of f. It will come out much neater than may be apparent at first.

VI, §4. RATIONAL FUNCTIONS

We shall now consider quotients of polynomials.

Example. Sketch the graph of the curve

$$y = f(x) = \frac{x-1}{x+1}$$

and determine the eight properties stated in the introduction.

1. When $x = 0$, we have $f(x) = -1$. When $x = 1$, $f(x) = 0$.
2. The derivative is

$$f'(x) = \frac{2}{(x+1)^2}.$$

(You can compute it using the quotient rule.) It is never 0, and therefore the function f has no critical points.

3. The denominator is a square and hence is always positive, wherever it is defined, that is for $x \neq -1$. Thus $f'(x) > 0$ for all $x \neq -1$. The function is increasing for all x, Of course, the function is not defined for $x = -1$ and neither is the derivative. Thus it would be more accurate to say that the function is increasing in the region

$$x < -1$$

and is increasing in the region $x > -1$.

4. There is no region of decrease.
5. Since the derivative is never 0, there is no relative maximum or minimum.

6. The second derivative is

$$f''(x) = \frac{-4}{(x+1)^3}.$$

There is no inflection point since $f''(x) \neq 0$ for all x where the function is defined. If $x < -1$, then the denominator $(x+1)^3$ is negative, and $f''(x) > 0$, so the graph is bending upward. If $x > -1$, then the denominator is positive, and $f''(x) < 0$ so the graph is bending downward.

7. As x becomes large positive, our function approaches 1 (using the method of §1). As x becomes large negative, our function also approaches 1.

There is one more useful piece of information which we can look into, when $f(x)$ itself becomes large positive or negative. This occurs near points where the denominator of $f(x)$ is 0. On the present instance, $x = -1$.

8. As x approaches -1, the denominator approaches 0 and the numerator approaches -2. If x approaches -1 from the right so $x > -1$, then the denominator is positive, and the numerator is negative. Hence the fraction

$$\frac{x-1}{x+1}$$

is negative, and is large negative.

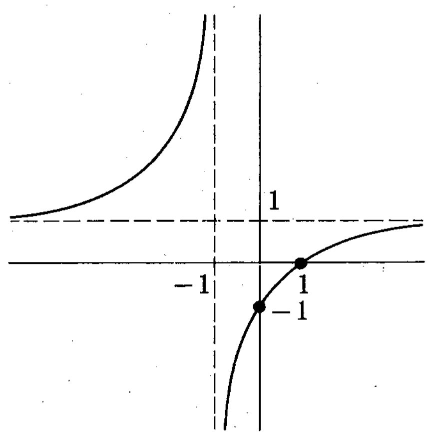


Figure 2

If x approaches -1 from the left so $x < -1$, then $x - 1$ is negative, but $x + 1$ is negative also. Hence $f(x)$ is positive and large, since the denominator is small when x is close to -1.

Putting all this information together, we see that the graph looks like that in the preceding figure.

We have drawn the two lines $x = -1$ and $y = 1$, as these play an important role when x approaches -1 and when x becomes large, positive or negative.

Remark. Again let

$$y = f(x) = \frac{x-1}{x+1}.$$

Then we can rewrite this relation to see directly that the graph of f is a hyperbola, as follows. We write the relation in the form

$$y = \frac{x+1-2}{x+1} = 1 - \frac{2}{x+1},$$

that is, $y - 1 = -2/(x + 1)$. Clearing denominators, this gives

$$(y-1)(x+1) = -2.$$

By Chapter II, you should know that this is a hyperbola. We worked out a sketch by a more general method, because it also works in cases when you cannot reduce the equation to one of the standard curves, like circles, parabolas, or hyperbolas.

Example. Sketch the graph of $f(x) = \dfrac{x^2 + x}{x - 1}$.

Note that f is not defined at $x = 1$. We can rewrite

$$f(x) = \frac{x(x+1)}{x-1}.$$

We have $f(x) = 0$ if and only if the numerator $x(x + 1) = 0$. Thus:

$$f(x) = 0 \quad \text{if and only if} \quad x = 0 \text{ or } x = -1.$$

Next we look at the derivative, which is

$$f'(x) = \frac{x^2 - 2x - 1}{(x-1)^2}.$$

(Compute it using the quotient rule.) Then

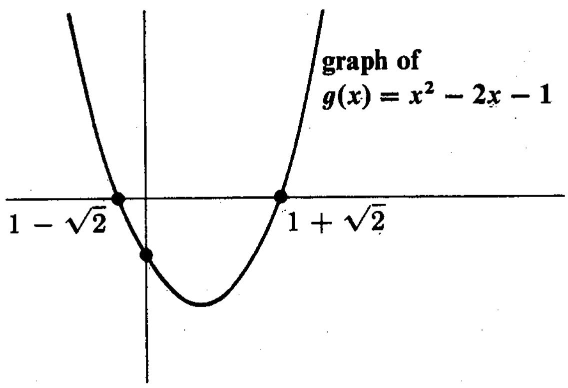
$$f'(x) = 0 \quad \Leftrightarrow \quad x^2 - 2x - 1 = 0,$$
$$\Leftrightarrow \quad x = 1 \pm \sqrt{2} \qquad \text{(by the quadratic formula)}$$

These are the critical points of f.

The denominator $(x-1)^2$ in $f'(x)$ is a square and hence is always positive, wherever it is defined, that is for $x \neq 1$. Therefore the sign of $f'(x)$ is the same as the sign of its numerator $x^2 - 2x - 1$. Let

$$g(x) = x^2 - 2x - 1.$$

The graph of g is a parabola, and since the coefficient of x^2 is $1 > 0$, this parabola is bending up as shown on the figure.



The two roots of $g(x) = 0$ are $x = 1 - \sqrt{2}$ and $1 + \sqrt{2}$. From the graph of $g(x)$ we see that

$$g(x) < 0 \qquad \text{when} \qquad 1 - \sqrt{2} < x < 1 + \sqrt{2},$$
$$g(x) > 0 \qquad \text{when} \qquad x < 1 - \sqrt{2} \text{ or } x > 1 + \sqrt{2}.$$

This gives us the regions of increase and decrease for $f(x)$.

For $x \leqq 1 - \sqrt{2}$, $f(x)$ is strictly increasing.

For $1 - \sqrt{2} \leqq x < 1$, $f(x)$ is strictly decreasing.

For $1 < x \leqq 1 + \sqrt{2}$, $f(x)$ is strictly decreasing.

For $1 + \sqrt{2} \leqq x$, $f(x)$ is strictly increasing.

It follows that f has a local maximum at $x = 1 - \sqrt{2}$ and f has a local minimum at $x = 1 + \sqrt{2}$.

As x becomes large positive, $f(x)$ becomes large positive as is seen from the expression

$$f(x) = \frac{x^2 + x}{x - 1} = \frac{x^2(1 + 1/x)}{x(1 - 1/x)} = x\,\frac{1 + 1/x}{1 - 1/x}.$$

As x becomes large negative, $f(x)$ becomes large negative.

As x approaches 1 and $x < 1$, the function $f(x)$ becomes large negative because the denominator $x - 1$ approaches 0, and is negative, while the numerator $x^2 + x$ approaches 2.

As x approaches 1 and $x > 1$, the function $f(x)$ becomes large positive because the denominator $x - 1$ approaches 0 and both numerator and denominator are positive, while the numerator approaches 2.

Hence the graph looks as drawn on Fig. 3.

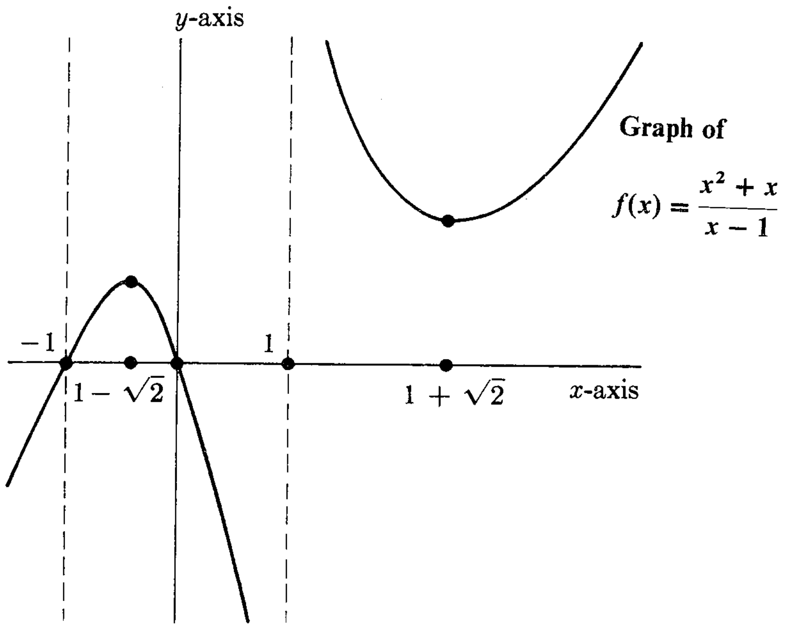


Figure 3

VI, §4. EXERCISES

Sketch the following curves, indicating all the information stated in the introduction. Regard convexity possibly as optional.

1. $y = \dfrac{x^2 + 2}{x - 3}$
2. $y = \dfrac{x - 3}{x^2 + 1}$
3. $y = \dfrac{x + 1}{x^2 + 1}$
4. $y = \dfrac{x^2 - 1}{x} = x - \dfrac{1}{x}$
5. $\dfrac{x}{x^3 - 1}$
6. $y = \dfrac{2x^2 - 1}{x^2 - 2}$
7. $y = \dfrac{2x - 3}{3x + 1}$
8. $\dfrac{4x}{x^2 - 9}$
9. $x + \dfrac{3}{x}$
10. $\dfrac{x^2 - 4}{x^3}$
11. $\dfrac{3x - 2}{2x + 3}$
12. $\dfrac{x}{3x - 5}$

13. $\dfrac{2x}{x+4}$ 14. $\dfrac{x^2}{\sqrt{x+1}}$ 15. $\dfrac{x+1}{x^2+5}$

16. $\dfrac{x+1}{x^2-5}$ 17. $\dfrac{x^2+1}{x^2-1}$ 18. $\dfrac{x^2-1}{x^2-4}$

19. Sketch the graph of $f(x) = x + 1/x$.

20. Let a, b be two positive numbers. Let

$$f(x) = ax + \frac{b}{x}.$$

Show that the minimum value of $f(x)$ for $x > 0$ is $2\sqrt{ab}$. Give reasons for your assertions. Deduce that $\sqrt{ab} \leqq (a+b)/2$. Sketch the graph of f for $x > 0$.

VI, §5. APPLIED MAXIMA AND MINIMA

This section deals with word problems concerning maxima and minima, and applies the techniques discussed previously. In each case, we want to maximize or minimize a function, which is at first given in terms of perhaps two variables. We proceed as follows.

1. Enough data is given so that one of these variables can be expressed in terms of the other, by some relation. We end up dealing with a function of only one variable.
2. We then find its critical points, setting the derivative equal to 0, and then determine whether the critical points are local maxima or minima.
3. We verify if these local maxima or minima are also maxima or minima for the whole interval of definition of the function. If the function is given only on some finite interval, it may happen that the maximum, say, occurs at an end point, where the derivative test does not apply.

Example 1. Find the point on the graph of the equation $y^2 = 4x$ which is nearest to the point (2, 3).

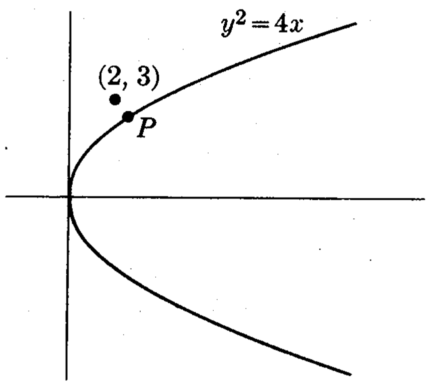


Figure 4

To minimize the distance between a point (x, y) and $(2, 3)$, it suffices to minimize the square of the distance, which has the advantage that no square root occurs in its formula. Indeed, suppose z_0^2 is a minimum value for the square of the distance, with z_0 positive. Then z_0 itself is a minimum value for the distance, because a positive number has a unique positive square root. The square of the distance is equal to

$$z^2 = (2 - x)^2 + (3 - y)^2.$$

Thus z^2 is expressed in terms of the two variables x, y. But we know that the point (x, y) lies on the curve whose equation is $y^2 = 4x$. Hence we can solve for one variable in terms of the other, namely $y = 2\sqrt{x}$. Substituting $y = 2\sqrt{x}$, we find an expression for the square of the distance only in terms of x, namely

$$\begin{aligned} f(x) &= (2 - x)^2 + (3 - 2\sqrt{x})^2 \\ &= 4 - 4x + x^2 + 9 - 12\sqrt{x} + 4x \\ &= 13 + x^2 - 12\sqrt{x}. \end{aligned}$$

We now determine the critical points of f. We have

$$f'(x) = 2x - \frac{6}{\sqrt{x}},$$

so

$$\begin{aligned} f'(x) = 0 \quad &\Leftrightarrow \quad 2x\sqrt{x} = 6 \\ &\Leftrightarrow \quad x = \sqrt[3]{9}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} f'(x) > 0 \quad &\Leftrightarrow \quad 2x\sqrt{x} > 6 \\ &\Leftrightarrow \quad x^3 > 9. \\ f'(x) < 0 \quad &\Leftrightarrow \quad x^3 < 9. \end{aligned}$$

Hence $f(x)$ is strictly increasing when $x > \sqrt[3]{9}$ and strictly decreasing when $x < \sqrt[3]{9}$. Hence $\sqrt[3]{9}$ is a minimum. When $x = \sqrt[3]{9}$ the corresponding value for y is

$$y = 2\sqrt{x} = 2\sqrt[3]{3}.$$

Hence the point on the graph of $y^2 = 4x$ closest to $(2, 3)$ is the point

$$P = (\sqrt[3]{9}, 2\sqrt[3]{3}).$$

Example 2. An oil can is to be made in the form of a cylinder to contain one quart of oil. What dimensions should it have so that the surface area is minimal (in other words, minimize the cost of material to make the can)?

Let r be the radius of the base of the cylinder and let h be its height. Then the volume is

$$V = \pi r^2 h.$$

The total surface area is the sum of the top, bottom, and circular sides, namely

$$A = 2\pi r^2 + 2\pi r h.$$

Thus the area is given in terms of the two variables r and h. However, we are also given that the volume V is constant, $V = 1$. Thus we get a relation between r and h,

$$\pi r^2 h = 1,$$

and we can solve for h in terms of r, namely

$$h = 1/\pi r^2.$$

Hence the area can be expressed entirely in terms of r, that is

$$A = 2\pi r^2 + 2\pi r/\pi r^2 = 2\pi r^2 + 2/r.$$

We want the area to be minimum. We first find the critical points of A. We have:

$$\begin{aligned} A'(r) = 4\pi r - 2/r^2 = 0 \quad &\Leftrightarrow \quad 4\pi r = 2/r^2 \\ &\Leftrightarrow \quad \pi r^3 = \tfrac{1}{2}. \end{aligned}$$

Thus we find exactly one critical point

$$r = \left(\frac{1}{2\pi}\right)^{1/3}$$

By physical considerations, we could see that this corresponds to a minimum, but we can also argue as follows. When r becomes large positive, or when r approaches 0, the function $A(r)$ becomes large, and so there has to be a minimum of the function for some value $r > 0$. This minimum is a critical point, and we have found that there is only one critical point. Hence we have found that the minimum occurs when r is the critical point. In this case we can solve back for h, namely

$$h = \frac{1}{\pi r^2} = \frac{(2\pi)^{2/3}}{\pi} = \frac{2^{2/3}}{\pi^{1/3}}.$$

This gives us the required dimensions.

Example 3. A truck is to be driven 200 mi at constant speed x mph. Speed laws require $30 \leqq x \leqq 60$. Assume that gasoline costs 50 cents/gallon and is consumed at the rate of

$$3 + \frac{x^2}{500} \text{ gal/hr.}$$

If the driver's wages are \$8 per hour, find the most economical speed.

We express the total cost as a sum of the cost of gasoline and the wages. The total time taken for the trip will be

$$\frac{200}{x}$$

because (time)(speed) = (distance) if the speed is constant. The cost of gas is then equal to the product of

(price per gallon)(number of gallons used per hr)(total time)

so that the cost of gasoline is

$$G(x) = \frac{1}{2}\left(3 + \frac{x^2}{500}\right)\frac{200}{x}.$$

(We write 1/2 because 50 cents $= \frac{1}{2}$ dollar.) On the other hand, the wages are given by the product

(wage per hour)(total time),

so that the cost of wages is

$$W(x) = 8 \cdot \frac{200}{x}.$$

Hence the total cost of the trip is

$$\begin{aligned} f(x) &= G(x) + W(x) \\ &= \frac{1}{2}\left(3 + \frac{x^2}{500}\right)\frac{200}{x} + \frac{8 \cdot 200}{x} \\ &= 100\left(\frac{3}{x} + \frac{x}{500}\right) + \frac{1600}{x}. \end{aligned}$$

We have

$$\begin{aligned} f'(x) &= -\frac{300}{x^2} + \frac{1}{5} - \frac{1600}{x^2} \\ &= -\frac{1900}{x^2} + \frac{1}{5}. \end{aligned}$$

Therefore $f'(x) = 0$ if and only if

$$\frac{1900}{x^2} = \frac{1}{5}$$

or in other words,

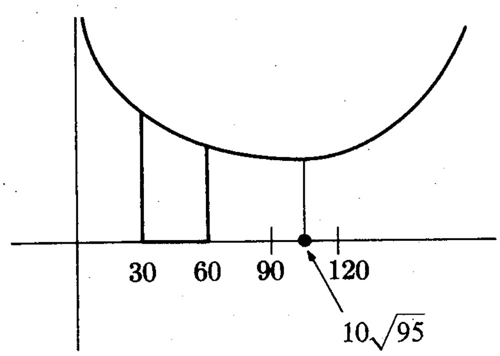
$$x^2 = 9500.$$

Thus $x = 10\sqrt{95}$. [We take x positive since this is the solution which has physical significance.]

Now we observe that $10\sqrt{95}$ is approximately equal to 10×10, and in any case is >60, so is beyond the speed limit of 60 which was assigned to begin with. Furthermore, if $0 < x < 10\sqrt{95}$ then

$$f'(x) < 0.$$

Hence the function $f(x)$ is decreasing for $0 \leqq x \leqq 10\sqrt{95}$. Its graph may be sketched as on the figure.



Since to begin with we restricted the possible speed to the interval $30 \leqq x \leqq 60$, it follows that the minimum of f over this interval must occur when $x = 60$. This is therefore the speed which minimizes the total cost.

Example 4. In the preceding example, suppose there is no speed limit. Then we see that if $x > 10\sqrt{95}$ then $f'(x) > 0$ so that $f(x)$ is increasing for

$$x > 10\sqrt{95}.$$

Therefore $10\sqrt{95}$ is a minimum point for f when no restriction is placed on x. Consequently, in this case, the speed which minimizes the cost is

$$x = 10\sqrt{95}.$$

Example 5. When light from a point source strikes a plane surface, the intensity of illumination is proportional to the cosine of the angle of incidence, and inversely proportional to the square of the distance from the source. How high should a light be located above the center of a circle of radius 12 ft to give the best illumination along the circumference?

The **angle of incidence** is measured from the perpendicular to the plane. The picture is as follows.

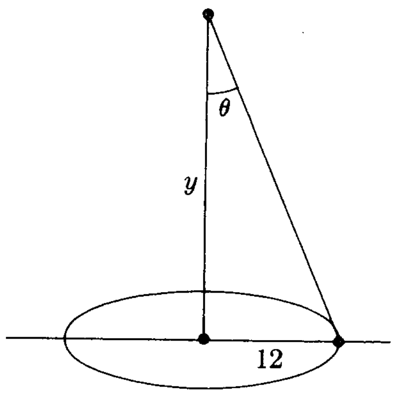


Figure 5

We denote by θ the angle of incidence, and by y the height of the light. Let I be the intensity of illumination. Two quantities are proportional means that there is a constant such that one is equal to the constant times the other. Thus there is a constant c such that

$$\begin{aligned} I(y) &= c \cos \theta \frac{1}{12^2 + y^2} \\ &= c \frac{y}{\sqrt{12^2 + y^2}} \frac{1}{12^2 + y^2} \\ &= \frac{cy}{(12^2 + y^2)^{3/2}}. \end{aligned}$$

The critical points of $I(y)$ are those points where $I'(y) = 0$. We have:

$$I'(y) = c\left[\frac{(12^2 + y^2)^{3/2} - y \cdot \frac{3}{2}(12^2 + y^2)^{1/2}(2y)}{(12^2 + y^2)^3}\right],$$

and this expression is equal to 0 precisely when the numerator is equal to 0, that is,

$$(12^2 + y^2)^{3/2} = 3y^2(12^2 + y^2)^{1/2}.$$

Canceling $(12^2 + y^2)^{1/2}$, we see that this is equivalent to

$$12^2 + y^2 = 3y^2,$$

or in other words,

$$12^2 = 2y^2.$$

Solving for y yields

$$y = \pm \frac{12}{\sqrt{2}}.$$

Only the positive value of y has physical significance, and thus the height giving maximum intensity is $12/\sqrt{2}$ ft, provided that we know that this critical point is a maximum for the function $I(y)$, for $y > 0$. This can be seen as follows.

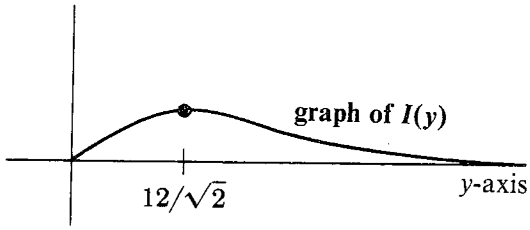
If y is very close to 0, then the numerator cy of $I(y)$ is close to 0, and the denominator $(12^2 + y^2)^{3/2}$ is close to $(12^2)^{3/2}$ so $I(y)$ is analyzed by factoring out y^2, namely

$$(12^2 + y^2)^{3/2} = \left(\frac{12^2}{y^2} + 1\right)^{3/2} y^3.$$

Therefore $I(y)$ tends to 0 as y becomes large, because

$$I(y) = \frac{cy}{(\text{term near } 1)y^3} = \frac{c}{(\text{term near } 1)} \frac{1}{y^2}$$

if y is large positive. Hence we have shown that $I(y)$ tends to 0 when y approaches 0 or y becomes large. It follows that $I(y)$ reaches a maximum for some value of $y > 0$, and this maximum must be a critical point. On the other hand, we have also proved that there is only one critical point. Hence this critical point is the maximum, as desired. Thus $y = 12/\sqrt{2}$ is a maximum for the function. In view of the preceding discussion, the graph can be sketched as on the figure.



Example 6. A business makes automobile transmissions selling for \$400. The total cost of marketing x units is

$$f(x) = 0.02x^2 + 160x + 400{,}000.$$

How many transmissions should be sold for maximum profit?

Let $P(x)$ be the profit coming from selling x units. Then $P(x)$ is the difference between the total receipts and the cost of marketing. Hence

$$\begin{aligned} P(x) &= 400x - (0.02x^2 + 160x + 400{,}000) \\ &= -0.02x^2 + 240x - 400{,}000. \end{aligned}$$

We want to know when $P(x)$ is maximum. We have:

$$P'(x) = -0.04x + 240$$

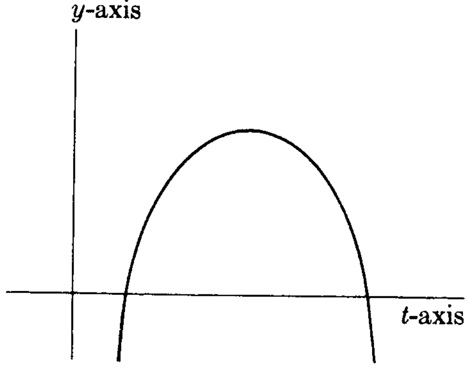
so the derivative is 0 when

$$0.04x = 240,$$

or in other words,

$$x = \frac{240}{0.04} = 6{,}000.$$

The equation $y = P(x)$ is a parabola, which bends down because the leading coefficient is -0.02 (negative). Hence the critical point is a maximum, and the answer is therefore 6,000 units.



Parabola $y = at^2 + bt + c$ with $a < 0$.

Example 7. A farmer buys a bull weighing 600 lbs, at a cost of \$180. It costs 15 cents per day to feed the animal, which gains 1 lb per day. Every day that the bull is kept, the sale price per pound declines according to the formula

$$B(t) = 0.45 - 0.00025t$$

where t is the number of days. How long should the farmer wait to maximize profits?

To do this, note that the total cost after time t is given by

$$f(t) = 180 + 0.15t.$$

The total sales amount to the product of the price $B(t)$ per pound times the weight of the animal, in other words,

$$\begin{aligned} S(t) &= (0.45 - 0.00025t)(600 + t) \\ &= -0.00025t^2 + 0.30t + 270. \end{aligned}$$

Hence the profit is

$$\begin{aligned} P(t) &= S(t) - f(t) \\ &= -0.00025t^2 + 0.15t + 90. \end{aligned}$$

Therefore

$$P'(t) = -0.0005t + 0.15$$

and $P'(t) = 0$ exactly when $0.0005t = 0.15$, or in other words,

$$t = \frac{0.15}{0.0005} = 300.$$

Hence the answer is 300 days for the farmer to wait before selling the bull, provided we can show that this value of t gives a maximum. But the formula for the profit is a quadratic expression in t, of the form

$$P(t) = at^2 + bt + c,$$

and $a < 0$. Hence $P(t)$ is a parabola, and since $a < 0$ this parabola opens downward as on the figure. Hence the critical point must be a maximum, as desired.

VI, §5. EXERCISES

1. Find the length of the sides of a rectangle of largest area which can be inscribed in a semicircle, the lower base being on the diameter.

2. A rectangular box has a square base and no top. The combined area of the sides and bottom is 48 ft^2. Find the dimensions of the box of maximum volume meeting these requirements.

3. Prove that, among all rectangles of given area, the square has the least perimeter.

4. A truck is to be driven 300 km at a constant speed of x km/hr. Speed laws require $30 \leqq x \leqq 60$. Assume that gasoline costs 30 cents/gallon and is consumed at the rate of $2 + x^2/600$ gal/hr. If the driver's wages are D dollars per hour, find the most economical speed and the cost of the trip if (a) $D = 0$, (b) $D = 1$, (c) $D = 2$, (d) $D = 3$, (e) $D = 4$.

5. A rectangle is to have an area of 64 m^2. Find its dimensions so that the distance from one corner to the mid-point of a non-adjacent side shall be a minimum.

6. Express the number 4 as the sum of two positive numbers in such a way that the sum of the square of the first and the cube of the second is as small as possible.

7. A wire 24 cm long is cut in two, and one part is bent into the shape of a circle, and the other into the shape of a square. How should it be cut if the sum of the areas of the circle and the square is to be (a) minimum, (b) maximum?

8. Find the point on the graph of the equation $y^2 = 4x$ which is nearest to the point (2, 1).

9. Find the points on the hyperbola $x^2 - y^2 = 1$ nearest to the point (0, 1).

10. Show that (2, 2) is the point on the graph of the equation $y = x^3 - 3x$ that is nearest the point (11, 1).

11. Find the coordinates of the points on the curve $x^2 - y^2 = 16$ which are nearest to the point (0, 6).

12. Find the coordinates of the points on the curve $y^2 = x + 1$ which are nearest to the origin.

13. Find the coordinates of the point on the curve $y^2 = \frac{5}{2}(x + 1)$ which is nearest to the origin.

14. Find the coordinates of the points on the curve $y = 2x^2$ which are closest to the point (9, 0).

15. A circular ring of radius b is uniformly charged with electricity, the total charge being Q. The force exerted by this charge on a particle at a distance x from the center of the ring, in a direction perpendicular to the plane of the ring, is given by $F(x) = Qx(x^2 + b^2)^{-3/2}$. Find the maximum of F for all $x \geqq 0$.

16. Let F be the rate of flow of water over a certain spillway. Assume that F is proportional to $y(h - y)^{1/2}$, where y is the depth of the flow, and h is the height, and is constant. What value of y makes F a maximum?

17. Find the point on the x-axis the sum of whose distances from (2, 0) and (0, 3) is a minimum.

18. A piece of wire of length L is cut into two parts, one of which is bent into the shape of an equilateral triangle and the other into the shape of a circle. How should the wire be cut so that the sum of the enclosed areas is:
(a) a minimum, (b) a maximum?

19. A fence $13\frac{1}{2}$ ft high is 4 ft from the side wall of a house. What is the length of the shortest ladder, one end of which will rest on the level ground outside the fence and the other on the side wall of the house?

20. A tank is to have a given volume V and is to be made in the form of a right circular cylinder with hemispheres attached to each end. The material for the ends costs twice as much per square meter as that for the sides. Find the most economical proportions. [You may assume that the area of a sphere is $4\pi r^2$.]

21. Find the length of the longest rod which can be carried horizontally around a corner from a corridor 8 ft wide into one 4 ft wide.

22. Let P, Q be two points in the plane on the same side of the x-axis. Let R be a point on the x-axis (Fig. 6). Show that the sum of the distances PR and QR is smallest when the angles θ_1 and θ_2 are equal.

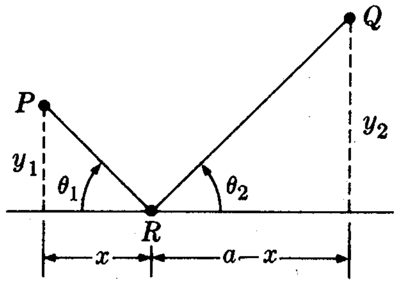


Figure 6

[*Hint*: First use the Pythagoras theorem to give an expression for the distances PR and RQ in terms of x and the fixed quantities y_1, y_2. Let $f(x)$ be the sum of the distances. Show that the condition $f'(x) = 0$ means that $\cos\theta_1 = \cos\theta_2$. Using values of x near 0 and a, show that $f(x)$ is decreasing near $x = 0$ and increasing near $x = a$. Hence the minimum must be in the open interval $0 < x < a$, and is therefore the critical point.]

23. Suppose the velocity of light is v_1 in air and v_2 in water. A ray of light traveling from a point P_1 above the surface of water to a point P_2 below the surface will travel by the path which requires the least time. Show that the ray will cross the surface at the point Q in the vertical plane through P_1 and P_2 so placed that

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2},$$

where θ_1 and θ_2 are the angles shown in the following figure:

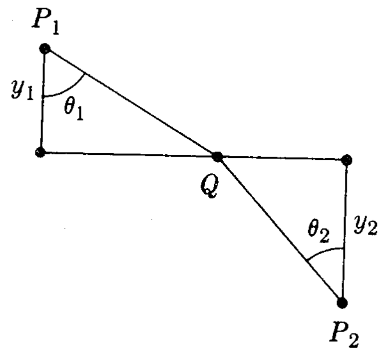


Figure 7

(You may assume that the light will travel in the vertical plane through P_1 and P_2. You may also assume that when the velocity is constant, equal to v, throughout a region, and s is the distance traveled, then the time t is equal to $t = s/v$.)

24. Let p be the probability that a certain event will occur, at any trial. In n trials, suppose that s successes have been observed. The likelihood function L is defined as $L(p) = p^s(1 - p)^{n-s}$. Find the value of p which maximizes the likelihood function. (Take $0 \leqq p \leqq 1$.) View n, s as constants.

25. Find an equation for the line through the following points making with the coordinate axes a triangle of minimum area in the first quadrant:
    (a) through the point (3, 1).
    (b) through the point (3, 2).

26. Let $a_1, \ldots, a_n$ be numbers. Show that there is a single number x such that

$$(x - a_1)^2 + \cdots + (x - a_n)^2$$

is a minimum, and find this number.

27. When light from a point source strikes a plane surface, the intensity of illumination is proportional to the cosine of the angle of incidence and inversely proportional to the square of the distance from the source. How high should a light be located above the center of a circle of radius 25 cm to give the best illumination along the circumference? (The angle of incidence is measured from the perpendicular to the plane.)

28. A horizontal reservoir has a cross section which is an inverted isosceles triangle, where the length of a leg is 60 ft. Find the angle between the equal legs to give maximum capacity.

29. A reservoir has a horizontal plane bottom and a cross section as shown on the figure. Find the angle of inclination of the sides from the horizontal to give maximum capacity.

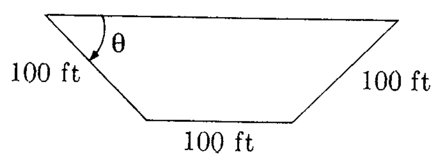


Figure 8

30. Determine the constant a such that the function

$$f(x) = x^2 + \frac{a}{x}$$

has (a) a local minimum at $x = 2$, (b) a local minimum at $x = -3$. (c) Show that the function cannot have a local maximum for any value of a.

31. The intensity of illumination at any point is proportional to the strength of the light source and varies inversely as the square of the distance from the source. If two sources of strengths a and b respectively are a distance c apart, at what point on the line joining them will the intensity be a minimum?

32. A window is in the shape of a rectangle surmounted by a semicircle. Find the dimensions when the perimeter is 12 ft and the area is as large as possible.

33. Find the radius and angle of the circular sector of maximum area if the perimeter is
(a) 20 cm (b) 16 cm.

34. You are watering the lawn and aiming the hose upward at an angle of inclination θ. Let r be the range of the hose, that is, the distance from the hose to the point of impact of the water. Then r is given by

$$r = \frac{2v^2}{g} \sin\theta\cos\theta,$$

where v, g are constants. For what angle is the range a maximum?

35. A ladder is to reach over a fence 12 ft high to a wall 2 ft behind the fence. What is the length of the shortest ladder that can be used?

36. A cylindrical tank is supposed to have a given volume V. Find the dimension of the radius of the base and the height in terms of V so that the surface area is minimal. The tank should be open on top, but closed at the bottom.

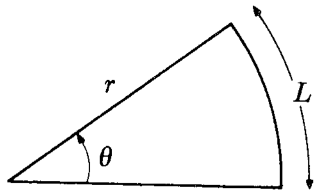
37. A flower bed is to have the shape of a circular sector of radius r and central angle θ. Find r and θ if the area is fixed and the perimeter is a minimum in case:
(a) $0 < \theta \leqq \pi$ and (b) $0 < \theta \leqq \pi/2$.

Recall that the area of a sector is

$$A = \pi r^2 \cdot \frac{\theta}{2\pi} = \theta r^2/2.$$

The length of an arc of a circle of radius r is

$$L = 2\pi r \cdot \frac{\theta}{2\pi} = r\theta.$$



38. A firm sells a product at $50 per unit. The total cost of marketing x units is given by the function

$$f(x) = 5000 + 650x - 45x^2 + x^3.$$

How many units should be produced per day to maximize profits? What is the daily profit for this number of units?

39. The daily cost of producing x units of a product is given by the formula

$$f(x) = 2002 + 120x - 5x^2 + \tfrac{1}{3}x^3.$$

Each unit sells for $264. How many units should be produced per day to maximize profits? What is the daily profit for this number of units?

40. A product is marketed at 50 dollars per unit. The total cost of marketing x units of the product is

$$f(x) = 1000 + 150x - 100x^2 + 2x^3.$$

How many units should be produced to maximize the profit? What is the daily profit for this x?

41. A company is the sole producer of a product, whose cost function is

$$f(x) = 100 + 20x + 2x^2.$$

If the company increases the price, then fewer units are sold, and in fact if we express the price $p(x)$ as a function of the number of units x, then

$$p(x) = 620 - 8x.$$

How many units should be produced to maximize profits? What is this maximum profit? [*Hint*: The total revenue is equal to the product $xp(x)$, number of units times the price.]

42. The cost of producing x units of a product is given by the function

$$f(x) = 10x^2 + 200x + 6{,}000.$$

If p is the price per unit, then the number of units sold at that price is given by

$$x = \frac{1000 - p}{10}.$$

For what value of x will the profit be positive? How many units should be produced to give maximum profits?