Nguyen Song Ha (Homework 10 - Calculus I) 1/3

Excellent

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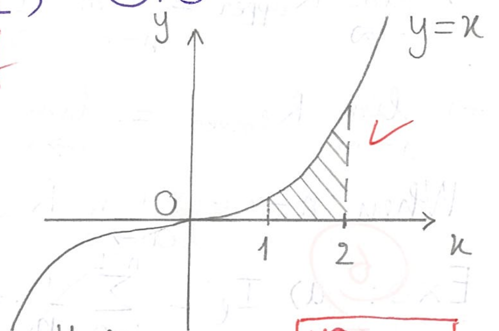
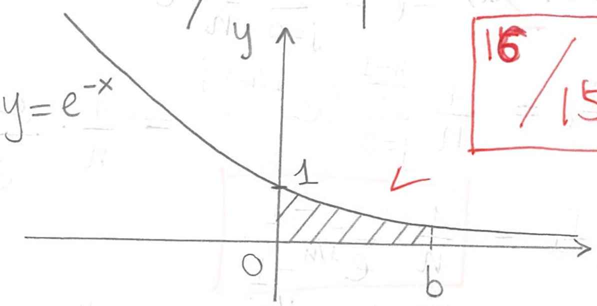
Ex1: 6 a) $x \to x^3$ between $x=1$ and $x=2$



$$\Rightarrow S = \int_1^2 x^3 dx = \frac{x^4}{4}\Big|_1^2 = \frac{2^4}{4} - \frac{1^4}{4} = \frac{15}{4}$$

b) $x \to e^{-x}$ between $x=0$ and $x=b>0$

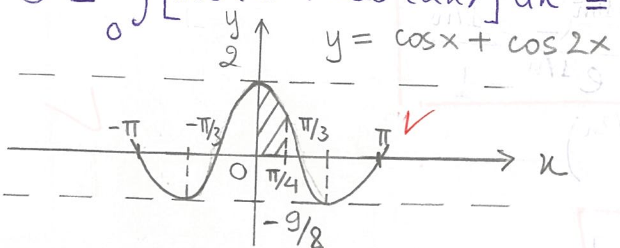


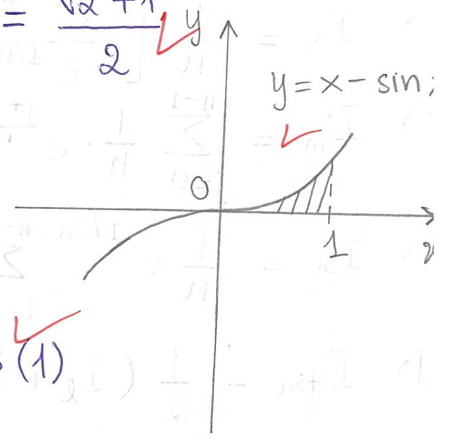
$$\Rightarrow S = \int_0^b e^{-x} = (-e^{-x})\Big|_0^b = 1 - e^{-b}$$

$$\Rightarrow \lim_{b\to\infty} S = \lim_{b\to\infty}(1-e^{-b}) = 1$$

c) $x \to \cos(x) + \cos(2x)$ between $x=0$ and $x=\frac{\pi}{4}$

$$\Rightarrow S = \int_0^{\pi/4} [\cos(x) + \cos(2x)]\, dx = \left[\sin(x) + \frac{\sin(2x)}{2}\right]\Big|_0^{\pi/4} = \frac{\sqrt{2}+1}{2}$$





d) $x \to x - \sin(x)$ between $x=0$ and $x=1$

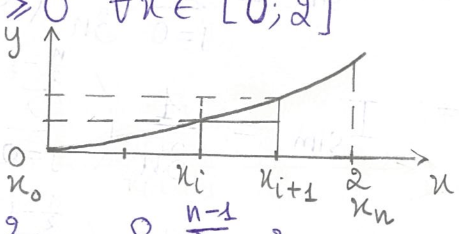
$$\Rightarrow S = \int_0^1 (x - \sin(x))\,dx = \left[\frac{x^2}{2} + \cos(x)\right]\Big|_0^1 = \frac{-1}{2} + \cos(1)$$

Ex2: 3 Regular partition of the interval divided into n intervals of the same length $\Rightarrow$ The length of each interval is $\Delta x_i = \frac{2}{n}$

Consider an interval $[x_i ; x_{i+1}]$ $(i = 0; 1; 2; \dots ; n-1)$

Set $m_i = \inf_{x\in[x_i;x_{i+1}]} f(x)$ | $f: \mathbb{R} \ni x \to x^2 \in \mathbb{R}$

$M_i = \sup_{x\in[x_i;x_{i+1}]} f(x)$ | $f'(x) = 2x \geq 0 \quad \forall x \in [0;2]$



$\Rightarrow$ On the interval $[0;2]$ $f(x)$ is increasing

$\Rightarrow m_i = f(x_i) = x_i^2$; $M_i = f(x_{i+1}) = x_{i+1}^2$

$$x_i = \left(\frac{2}{n}\right)i \Rightarrow R_{lower} = \sum_{i=0}^{n-1} m_i \Delta x_i = \sum_{i=0}^{n-1}\left(\frac{2i}{n}\right)^2 \cdot \frac{2}{n} = \frac{8}{n^3}\sum_{i=0}^{n-1} i^2$$

$$\Rightarrow R_{lower} = \frac{8}{n^3}\cdot\frac{(n-1)n(2n-1)}{6} = \frac{4}{3}\cdot\frac{(n-1)(2n-1)}{n^2}$$

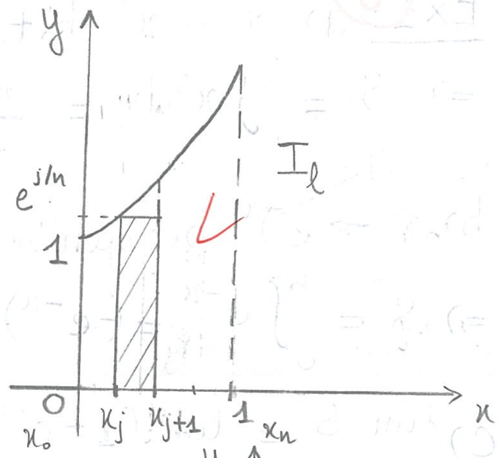
$$\lim_{n\to\infty} R_{lower} = \lim_{n\to\infty}\frac{4}{3}\cdot\frac{(n-1)(2n-1)}{n^2} = \lim_{n\to\infty}\frac{4}{3}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) = \frac{8}{3}$$

$$R_{upper} = \sum_{i=0}^{n-1} M_i \Delta x_i = \sum_{i=0}^{n-1}\left(\frac{2(i+1)}{n}\right)^2\cdot\frac{2}{n} = \sum_{j=1}^{n}\frac{8}{n^3}\cdot j^2 = \frac{4}{3}\cdot\frac{(n+1)(2n+1)}{n^2}$$

$$\Rightarrow \lim_{n\to\infty} R_{upper} = \lim_{n\to\infty} \frac{4}{3}\cdot\frac{(n+1)(2n+1)}{n^2} = \lim_{n\to\infty}\frac{4}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{8}{3}$$

$$\Rightarrow \lim_{n\to\infty} R_{lower} = \lim_{n\to\infty} R_{upper} = \frac{8}{3}$$

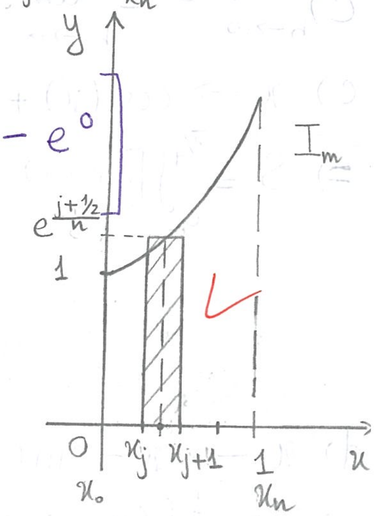
When $n\to\infty$: $\lim_{n\to\infty} R_{upper} = \lim_{n\to\infty} R_{lower} = \frac{8}{3}$



Ex3: a) $I_l = \sum_{j=0}^{n-1}\frac{1}{n}\cdot e^{j/n}$

$$\Rightarrow I_l = \frac{1}{n}\sum_{j=0}^{n-1}(e^{1/n})^j = \frac{1}{n}\cdot\frac{(e^{1/n})^n-1}{e^{1/n}-1}$$

$$\Rightarrow I_l = \boxed{\frac{1}{n}\cdot\frac{e-1}{e^{1/n}-1}}$$

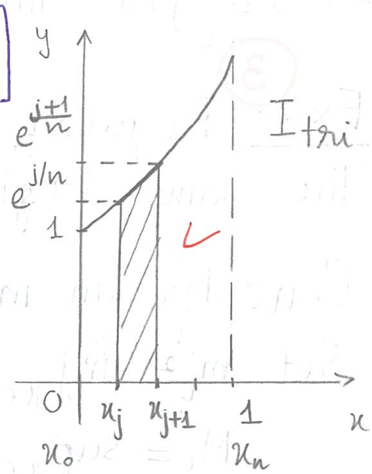


b) $I_r = \sum_{j=1}^{n}\frac{1}{n}\cdot e^{j/n} = \frac{1}{n}\sum_{j=1}^{n}(e^{1/n})^j = \frac{1}{n}\cdot\left[\frac{(e^{1/n})^{n+1}-1}{e^{1/n}-1} - e^0\right]$

$$\Rightarrow I_r = \frac{1}{n}\left[\frac{e^{(n+1)/n}-1}{e^{1/n}-1}-1\right] = \boxed{\frac{1}{n}\cdot\frac{e^{1/n}(e-1)}{e^{1/n}-1}}$$

c) $I_m = \sum_{j=0}^{n-1}\frac{1}{n}\cdot e^{\frac{j+1/2}{n}} = \frac{1}{n}\sum_{j=0}^{n-1}\left(e^{j/n}\cdot e^{1/2n}\right)$

$$\Rightarrow I_m = \frac{1}{n}e^{1/2n}\sum_{j=0}^{n-1}(e^{1/n})^j = \boxed{\frac{1}{n}e^{1/2n}\cdot\frac{e-1}{e^{1/n}-1}}$$



d) $I_{tri} = \frac{1}{2}(I_l+I_r) = \frac{1}{2}\left[\frac{e-1}{n(e^{1/n}-1)}+\frac{e^{(1+1/n)}-e^{1/n}}{n(e^{1/n}-1)}\right]$

$$\Rightarrow I_{tri} = \frac{1}{2n(e^{1/n}-1)}\cdot\left(e-1+e^{1/n}(e-1)\right)$$

$$\Rightarrow I_{tri} = \boxed{\frac{(e-1)(e^{1/n}+1)}{2n(e^{1/n}-1)}}$$

e) $I_{sim} = \sum_{j=0}^{\frac{n}{2}-1}\frac{1}{3n}\left(e^{\frac{2j}{n}}+4e^{\frac{2j+1}{n}}+e^{\frac{2j+2}{n}}\right)$ for n even

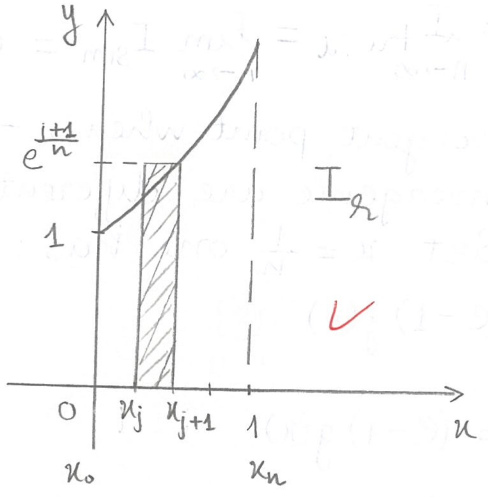
$$\Rightarrow I_{sim} = \frac{1}{3n}\left(\sum_{j=0}^{\frac{n}{2}-1}e^{2j/n}+\sum_{j=0}^{\frac{n}{2}-1}4e^{\frac{2j+1}{n}}+\sum_{j=0}^{\frac{n}{2}-1}e^{\frac{2j+2}{n}}\right)$$

$$\Rightarrow I_{sim} = \frac{1}{3n}\left(\sum_{j=0}^{\frac{n}{2}-1}(e^{2/n})^j+4\sum_{j=0}^{\frac{n}{2}-1}(e^{2/n})^j\cdot e^{1/n}+\sum_{j=0}^{\frac{n}{2}-1}(e^{2/n})^j\cdot e^{2/n}\right)$$

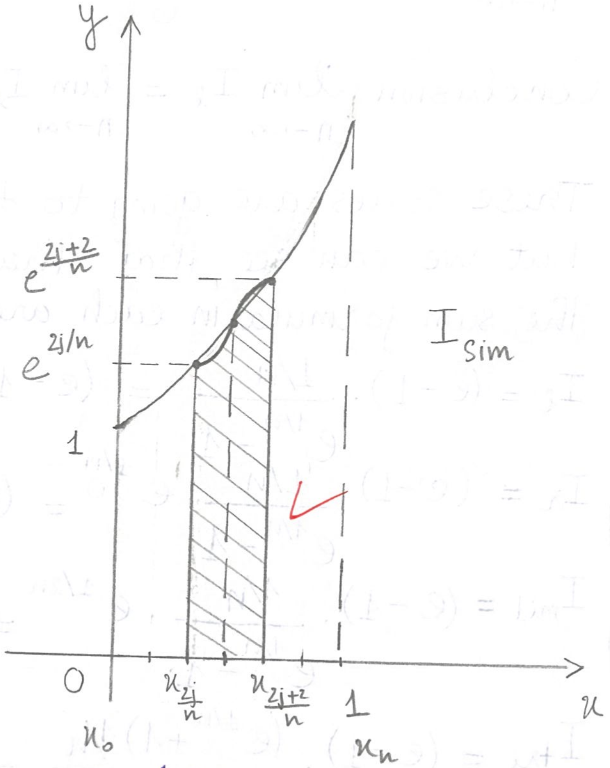
$$\Rightarrow I_{sim} = \frac{1}{3n}\left(\frac{(e^{2/n})^{\frac{n}{2}}-1}{e^{2/n}-1}+4e^{1/n}\cdot\frac{(e^{2/n})^{\frac{n}{2}}-1}{e^{2/n}-1}+e^{2/n}\cdot\frac{(e^{2/n})^{\frac{n}{2}}-1}{e^{2/n}-1}\right)$$

$$\Rightarrow I_{sim} = \boxed{\frac{1}{3n}\cdot\frac{e-1}{e^{2/n}-1}\left(1+4e^{1/n}+e^{2/n}\right)}$$

Ex3 : (continue)



(Right rule)



(Simpson's rule)

⊕ $\lim\limits_{n\to\infty} I_l = \lim\limits_{n\to\infty} \frac{1}{n}\cdot\frac{e-1}{e^{1/n}-1}$

$= \lim\limits_{t\to 0} (e-1)\cdot\frac{t}{e^t-1}$ (set $t = \frac{1}{n}$)

$= (e-1)\lim\limits_{t\to 0}\frac{t}{e^t-1} = (e-1)\lim\limits_{t\to 0}\frac{1}{e^t} = e-1$

(Using L'Hospital's rule since $\lim\limits_{t\to 0} t = \lim\limits_{t\to 0}(e^t-1) = 0$)

⊕ $\lim\limits_{n\to\infty} I_r = \lim\limits_{n\to\infty}\frac{1}{n}\cdot\frac{e-1}{e^{1/n}-1}\cdot e^{1/n} = \left(\lim\limits_{n\to\infty} I_l\right)\cdot\left(\lim\limits_{n\to\infty} e^{1/n}\right) = e-1$

⊕ $\lim\limits_{n\to\infty} I_m = \lim\limits_{n\to\infty}\frac{1}{n}\cdot\frac{e-1}{e^{1/n}-1}\cdot e^{1/2n} = \left(\lim\limits_{n\to\infty} I_l\right)\cdot\left(\lim\limits_{n\to\infty} e^{1/2n}\right) = e-1$

⊕ $\lim\limits_{n\to\infty} I_{tri} = \lim\limits_{n\to\infty}\frac{1}{n}\cdot\frac{e-1}{e^{1/n}-1}\cdot\frac{e^{1/n}+1}{2} = \left(\lim\limits_{n\to\infty} I_l\right)\cdot\left(\lim\limits_{n\to\infty}\frac{e^{1/n}+1}{2}\right)$

$= (e-1)\cdot\frac{1+1}{2} = e-1$

⊕ $\lim\limits_{n\to\infty} I_{sim} = \lim\limits_{n\to\infty}\frac{1}{n}\cdot\frac{(e-1)(1+4e^{1/n}+e^{2/n})}{3(e^{2/n}-1)}$

$= \lim\limits_{n\to\infty}\frac{1+4e^{1/n}+e^{2/n}}{6}\cdot\frac{(e-1)\frac{2}{n}}{e^{2/n}-1}$

$= \left[\lim\limits_{n\to\infty}\frac{1+4e^{1/n}+e^{2/n}}{6}(e-1)\right]\cdot\left(\lim\limits_{t\to 0}\frac{t}{e^t-1}\right)$ (Set $t = \frac{2}{n}$)

$$\Rightarrow \lim_{n\to\infty} I_{sim} = \frac{1+4+1}{6}(e-1).\lim_{t\to 0}\frac{1}{e^t} = e-1$$

Conclusion $\lim_{n\to\infty} I_l = \lim_{n\to\infty} I_r = \lim_{n\to\infty} I_m = \lim_{n\to\infty} I_{tri} = \lim_{n\to\infty} I_{sim} = e-1$

These series are going to the same convergent point when $n\to\infty$ but we can see that their rate of convergence are different since the sum formula in each are different. Set $x=\frac{1}{n}$ one has:

$$\begin{cases} I_l = (e-1).\dfrac{1/n}{e^{1/n}-1} = (e-1).\dfrac{x}{e^x-1} = (e-1)f(x) \\ I_r = (e-1).\dfrac{1/n}{e^{1/n}-1}.e^{1/n} = (e-1)\dfrac{xe^x}{e^x-1} = (e-1)g(x) \\ I_m = (e-1).\dfrac{1/n}{e^{1/n}-1}.e^{1/2n} = (e-1)\dfrac{xe^{x/2}}{e^x-1} = (e-1)h(x) \\ I_{tri} = (e-1).\dfrac{(e^{1/n}+1)1/n}{2(e^{1/n}-1)} = (e-1)\dfrac{x(e^x+1)}{2(e^x-1)} = (e-1)k(x) \\ I_{sim} = (e-1).\dfrac{1/n(1+4e^{1/n}+e^{2/n})}{3(e^{2/n}-1)} = (e-1).\dfrac{x(1+4e^x+e^{2x})}{3(e^{2x}-1)} = (e-1)l(x) \end{cases}$$

$$\Rightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} h(x) = \lim_{x\to 0^+} k(x) = \lim_{x\to 0^+} l(x) = 1$$

$$\Rightarrow \lim_{x\to 0^+}(f(x)-1) = \lim_{x\to 0^+}(g(x)-1) = \lim_{x\to 0^+}(h(x)-1) = \lim_{x\to 0^+}(k(x)-1) = \lim_{x\to 0^+}(l(x)-1) = 0$$

+ Consider: $\lim_{n\to\infty}\dfrac{I_l-(e-1)}{I_r-(e-1)} = \lim_{x\to 0^+}\dfrac{f(x)-1}{g(x)-1} = \lim_{x\to 0^+}\dfrac{x-e^x+1}{xe^x-e^x+1}$

$$\Rightarrow \lim_{n\to\infty}\frac{I_l-(e-1)}{I_r-(e-1)} = \lim_{x\to 0^+}\frac{1-e^x}{xe^x} = \lim_{x\to 0^+}\frac{-e^x}{e^x+xe^x} = -1 \quad (1)$$

(Using L'Hospital's rule)

+ Consider: $\lim_{n\to\infty}\dfrac{I_l-(e-1)}{I_m-(e-1)} = \lim_{x\to 0^+}\dfrac{f(x)-1}{h(x)-1} = \lim_{x\to 0^+}\dfrac{x-e^x+1}{xe^{x/2}-e^x+1}$

$$\Rightarrow \lim_{n\to\infty}\frac{I_l-(e-1)}{I_m-(e-1)} = \lim_{x\to 0^+}\frac{1-e^x}{e^{x/2}+\frac{xe^{x/2}}{2}-e^x} = \lim_{x\to 0^+}\frac{-e^x}{e^{x/2}+\frac{x}{4}e^{x/2}-e^x} = \infty \quad (2)$$

+ Consider: $\lim_{n\to\infty}\dfrac{I_l-(e-1)}{I_{tri}-(e-1)} = \lim_{x\to 0^+}\dfrac{f(x)-1}{k(x)-1} = \lim_{x\to 0^+}\dfrac{x-e^x+1}{(e^x+1)\frac{x}{2}-e^x+1}$

$$\Rightarrow \lim_{n\to\infty}\frac{I_l-(e-1)}{I_{tri}-(e-1)} = \lim_{x\to 0^+}\frac{(1-e^x).2}{xe^x+1-e^x} = \lim_{x\to 0^+}\frac{-2e^x}{xe^x} = -\infty \quad (3)$$

Ex3 : (continue)

⊕ Consider: $\lim\limits_{n\to\infty} \dfrac{I_{tri} - (e-1)}{I_m - (e-1)} = \lim\limits_{x\to 0^+} \dfrac{\frac{x}{2}(e^x+1) - e^x + 1}{xe^{x/2} - e^x + 1}$

$$\Rightarrow \lim_{n\to\infty} \frac{I_{tri} - (e-1)}{I_m - (e-1)} = \lim_{x\to 0^+} \frac{xe^x + 1 - e^x}{2e^{x/2} + xe^{x/2} - 2e^x}$$

$$= \lim_{x\to 0^+} \frac{e^x + xe^x - e^x}{2e^{x/2} + \frac{x}{2}e^{x/2} - 2e^x}$$

$$= \lim_{x\to 0^+} \frac{2(e^x + xe^x)}{3e^{x/2} + \frac{x}{2}e^{x/2} - 4e^x} = -2 \quad (4)$$

From (1); (2); (3); (4) we can see that the rate of convergence of I_l and I_r are equal ; the rate of I_{tri} and I_m are kind of equal and both are faster than the rate of I_l and I_r

Using the same method we can consider :

$$\lim_{n\to\infty} \frac{I_{sim} - (e-1)}{I_m - (e-1)} = \lim_{x\to 0^+} \frac{l(x) - 1}{h(x) - 1}$$

$$= \lim_{x\to 0^+} \frac{\frac{x}{3}(e^{2x} + 4e^x + 1) - e^{2x} + 1}{(xe^{x/2} - e^x + 1)(e^x + 1)}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{3}(e^{2x} + 4e^x + 1) + \frac{x}{3}(2e^{2x} + 4e^x) - 2e^{2x}}{e^{\frac{3x}{2}} + e^{\frac{x}{2}} + x\left(\frac{3}{2}e^{\frac{3x}{2}} + \frac{1}{2}e^{\frac{x}{2}}\right) - 2e^{2x}}$$

$$= \lim_{x\to 0^+} \frac{\frac{4}{3}(e^{2x} + 2e^x) + \frac{4x}{3}(e^{2x} + e^x) - 4e^{2x}}{3e^{\frac{3x}{2}} + e^{\frac{x}{2}} + \frac{x}{4}\left(9e^{\frac{3x}{2}} + e^{\frac{x}{2}}\right) - 4e^{2x}}$$

$$= \lim_{x\to 0^+} \frac{4(e^{2x} + e^x) + \frac{4x}{3}(2e^{2x} + e^x) - 8e^{2x}}{\frac{9}{2}e^{\frac{3x}{2}} + \frac{1}{2}e^{\frac{x}{2}} + \frac{1}{4}\left(\frac{27}{2}e^{\frac{3x}{2}} + \frac{1}{2}e^{\frac{x}{2}}\right)(x+1) - 8e^{2x}}$$

$$= 0$$

great!

$\Rightarrow$ The rate of convergence of I_{sim} is faster than the rate of I_m; I_{tri} and its rate is the fastest among I_l; I_r; I_m; I_{tri}; I_{sim}