Nagoya University, G30 program Fall 2016

Calculus I Instructor : Serge Richard

Midterm

Exercise 1 *Compute the following limits and explain your computations :*

1. $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$,
2. $\lim_{x\to 1-} \frac{(x-1)}{|x-1|} \mathrm{e}^{2x}$,
3. $\lim_{x\to 0} \frac{(\mathrm{e}^x-1)^3}{x^3}$,
4. $\lim_{x\to 0} \frac{(\mathrm{e}^x-1)^{25}}{x^{25}}$,

Exercise 2 *Compute the derivative of the following functions:*

1. $f : \mathbb{R}^* \to \mathbb{R},\ f(x) = \mathrm{e}^x \cos(1/x)$,
2. $g : \mathbb{R} \to \mathbb{R},\ g(x) = \frac{x^2+2}{x^2-2}$,
3. $h : \mathbb{R}^* \to \mathbb{R},\ h(x) = |x| \sin(x)$,

Exercise 3 *Sketch the following curve as precisely as possible:*

$$f : \mathbb{R} \ni x \mapsto \frac{x+1}{x^2+8} \in \mathbb{R}.$$

Exercise 4 *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Let $x_0 \in (a,b)$ be the global maximum of f on $[a,b]$. What can you say about $f'(x_0)$? Prove your statement.*

Exercise 5 *Consider the curve in $\mathbb{R}^2$ defined by the relation*

$$F(x,y) = 3x^3y - y^4 + 5x^2 + 5 = 0.$$

1. *Find the slope of the tangent at the point $(1,2)$,*
2. *Show that the points $\big(0, -\sqrt[4]{5}\big)$ and $\big(0, \sqrt[4]{5}\big)$ belong to the curve and that the tangent at these points is horizontal.*

Exercise 6 *For any $x \in \mathbb{R}$ with $x \neq -1$ we consider the sequence $(a_n)_{n\in\mathbb{N}}$ given by*

$$a_n := \frac{x^n}{1+x^n}.$$

For which x does the limit $\lim_{x\to\infty} a_n$ exists ? Give the value of this limit whenever it exists.

Total : 30 pts

Midterm

Exercise 1

6 pts (4 × 1 for answer, 4 × ½ for presentation)

1) $$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{2}.$$

Or by l'Hospital's rule :

$$\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{\frac{1}{2}\frac{1}{\sqrt{1+x}}}{1} = \lim_{x\to 0} \frac{1}{2}\frac{1}{\sqrt{1+x}} = \frac{1}{2}.$$

2) $$\lim_{x\to 1_-} \frac{x-1}{|x-1|} e^{2x} = \lim_{\substack{x\to 1\\ x<1}} \frac{x-1}{-(x-1)} e^{2x} = \lim_{\substack{x\to 1\\ x<1}} -e^{2x} = -e^2.$$

3) $$\lim_{x\to 0} \frac{(e^x-1)^3}{x^3} \overset{\text{Hosp.}}{=} \lim_{x\to 0} \frac{3(e^x-1)^2 e^x}{3x^2} \overset{\text{Hosp.}}{=} \lim_{x\to 0} \frac{2(e^x-1)(e^x)^2 + e^x(e^x-1)^2}{2x}$$

$$\overset{\text{Hosp.}}{=} \lim_{x\to 0} \frac{(e^x)^3 + (e^x-1)2(e^x)^2 + \frac{1}{2}e^x(e^x-1)^2 + 2(e^x-1)(e^x)^2}{1}$$

$$= 1.$$

4) One observes that $\lim_{x\to 0} \frac{(e^x-1)^n}{x^n} \overset{\text{Hosp}}{=} \lim_{x\to 0} \frac{(e^x-1)^{n-1} e^x}{x^{n-1}}$

$= \lim_{x\to 0} \frac{(e^x-1)^{n-1}}{x^{n-1}} \lim_{x\to 0} e^x$ if $\lim_{x\to 0} \frac{(e^x-1)^{n-1}}{x^{n-1}}$ exists.

But one easily shows that for $n=1$, one gets 1 or for $n=3$, one has shown that $\lim_{x\to 0} \frac{(e^x-1)^3}{x^3} = 1$.

Thus, such a result is true for any n, and in particular $\lim_{x\to 0} \frac{(e^x-1)^n}{x^n} = 1.$

Other method : $\lim_{x\to 0} \frac{(e^x-1)^n}{x^n} = \lim_{x\to 0} \left(\frac{e^x-1}{x}\right)^n = \left(\lim_{x\to 0} \frac{e^x-1}{x}\right)^n = (e^0)^n = 1^n = 1.$

Exercise 2 6 pts (3 × 2)

1) $f'(x) = e^x \cos(1/x) + e^x \sin(1/x) \frac{1}{x^2}$

$= e^x \left(\cos(1/x) + \frac{1}{x^2} \sin(1/x) \right).$

2) $g'(x) = \frac{2x(x^2-2) - 2x(x^2+2)}{(x^2-2)^2} = \frac{-8x}{(x^2-2)^2}.$

3) $h'(x) = \begin{cases} \sin(x) + x\cos(x) & \text{if } x > 0 \\ -\sin(x) - x\cos(x) & \text{if } x < 0 \end{cases}.$

Exercise 3 5 pts

1) $f(x) = 0 \iff x + 1 = 0 \iff x = -1$

$f(0) = \frac{1}{8}.$ 1

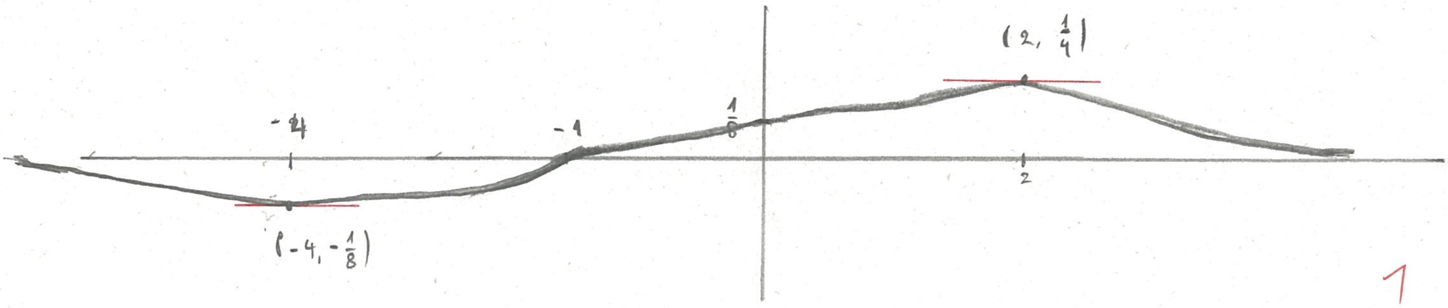
2) $f'(x) = \frac{x^2+8 - 2x(x+1)}{(x^2+8)^2} = \frac{-x^2 - 2x + 8}{(x^2+8)^2} = -\frac{x^2+2x-8}{(x^2+8)^2}$

$= -\frac{(x+4)(x-2)}{(x^2+8)^2}.$ 1

$f'(x) = 0 \iff x = -4$ or $x = 2$

Note that $f(2) = \frac{3}{12} = \frac{1}{4}$ and $f(-4) = \frac{-3}{24} = -\frac{1}{8}.$ 1

3) $\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{1 + 1/x}{x + 8/x} = 0.$ 1



1

Exercise 4 3 pts

$f(x_0)$ global maximum $\Rightarrow f'(x_0) = 0$. 1

Indeed for $h > 0$

$$0 \geq \frac{f(x_0+h) - f(x_0)}{h} \xrightarrow{h \to 0} f'(x_0)$$

but for $h < 0$

$$0 \leq \frac{f(x_0+h) - f(x_0)}{h} \xrightarrow{h \to 0} f'(x_0)$$

Thus, one deduces that $f'(x_0) \geq 0$ and $f'(x_0) \leq 0$ which implies $f'(x_0) = 0$.

2

Exercise 5 5 pts

1) If $y = y(x)$ one has

$$\frac{d}{dx} F(x, y(x)) = 9x^2 y + 3x^3 \frac{dy}{dx} - 4y^3 \frac{dy}{dx} + 10x = 0$$

$$\Leftrightarrow \quad -9x^2 y - 10x = (3x^3 - 4y^3) \frac{dy}{dx}$$

$$\Leftrightarrow \quad \frac{dy}{dx} = \frac{-9x^2 y - 10x}{3x^2 - 4y^3} = -\frac{x(9xy+10)}{3x^2 - 4y^3}.$$ 2

At $(1, 2)$ which satisfies $F(1,2) = 0$ one has

$$\frac{dy}{dx}(1) = -\frac{1(18+10)}{3-32} = \frac{28}{29}.$$ 1

2) One has $F(0, \pm\sqrt[4]{5}) = -(\pm\sqrt[4]{5})^4 + 5 = -5+5 = 0$

$\Rightarrow (0, \pm\sqrt[4]{5})$ belong to the curve. 1

Since $\frac{dy}{dx} = -\frac{x(9xy+10)}{3x^2 - 4y^3}$, at $(0, \pm\sqrt[4]{5})$ one gets

$\frac{dy}{dx}(0) = 0 \Rightarrow$ horizontal tangent. 1

Exercise 6   5 pts   (3 answers, 2 presentation)

$a_n = \frac{x^n}{1+x^n}$.

1) If $x = 1$, then $a_n = \frac{1}{1+1} = \frac{1}{2}$ $\forall n$

$\Rightarrow \lim_{n \to \infty} a_n = \frac{1}{2}$

2) If $|x| > 1$, then $a_n = \frac{1}{1 + 1/x^n} = \frac{1}{1 + (1/x)^n}$

with $-1 < \frac{1}{x} < 1$. Thus $\left(\frac{1}{x}\right)^n \xrightarrow{n \to \infty} 0$ and $\lim_{n \to \infty} a_n = 1$.

3) If $x \in [0,1)$ then $0 < a_n = \frac{x^n}{1+x^n} < x^n \xrightarrow[n \to \infty]{} 0$

Thus $\lim_{n \to \infty} a_n = 0$.

4) If $x \in (-1, 0)$, then $0 < |a_n| \leq \frac{|x|^n}{1-|x|} \xrightarrow{n \to \infty} 0$

Thus $\lim_{n \to \infty} a_n = 0$.