Homework 14 - Exercise 2

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$$I_\infty = \sum_{j=1}^{\infty} \frac{j}{3^j};$$

Compute: $$I_n = \sum_{j=1}^{n} \frac{j}{3^j}$$

$$I_n = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \cdots + \frac{n}{3^n}$$

$$= \left(\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}\right) + \left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n}\right) + \cdots$$

$$+ \left(\frac{1}{3^{n-1}} + \frac{1}{3^n}\right) + \left(\frac{1}{3^n}\right)$$

$$= \frac{\left(\frac{1}{3}\right)^{n+1} - \frac{1}{3}}{\frac{1}{3} - 1} + \frac{\left(\frac{1}{3}\right)^{n+1} - \left(\frac{1}{3}\right)^2}{\frac{1}{3} - 1} + \cdots + \frac{\left(\frac{1}{3}\right)^{n+1} - \left(\frac{1}{3}\right)^n}{\frac{1}{3} - 1}$$

$$= \frac{n\left(\frac{1}{3}\right)^{n+1} - \left(\frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^n}\right)}{\frac{1}{3} - 1}$$

$$= \frac{n\left(\frac{1}{3}\right)^{n+1} - \frac{\left(\frac{1}{3}\right)^{n+1} - \frac{1}{3}}{\frac{1}{3} - 1}}{\frac{1}{3} - 1}$$

$$= -\frac{3}{2}n\left(\frac{1}{3}\right)^{n+1} + \left(\frac{3}{2}\right)^2\left(\frac{1}{3} - \left(\frac{1}{3}\right)^{n+1}\right)$$

$$= \frac{9}{4}\left(\frac{1}{3} - \left(\frac{1}{3}\right)^{n+1}\right) - \frac{3}{2}n\left(\frac{1}{3}\right)^{n+1}$$

$$= \frac{3}{4} - \frac{9}{4}\left(\frac{1}{3}\right)^{n+1} - \frac{3}{2}n\left(\frac{1}{3}\right)^{n+1}$$

$$= \frac{3}{4} - \frac{3}{4}(3 + 2n)\left(\frac{1}{3}\right)^{n+1}$$

$$= \frac{3}{4}\left(1 - \frac{2n+3}{3^{n+1}}\right)$$

Hence: $$I_\infty = \lim_{n\to\infty} \sum_{j=1}^{n} \frac{j}{3^j} = \lim_{n\to\infty} \frac{3}{4}\left(1 - \frac{2n+3}{3^{n+1}}\right)$$

$$= \frac{3}{4} - \lim_{n\to\infty} \frac{2n+3}{3^{n+1}} \overset{\text{L'hospital}}{=} \frac{3}{4} - \lim_{n\to\infty} \frac{2}{3^{n+1}\ln 3} = \frac{3}{4}.$$