Nagoya University, G30 program Fall 2016

Mathematics Tutorial Ia (calculus) Instructor : Serge Richard

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Homework 9

Exercise 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be the unique differentiable function satisfying $f'(x) = f(x)$ for any $x \in \mathbb{R}$ and $f(0) = 1$. Show the following properties of this function :*

(i) $f(x)f(-x) = 1$ *for any $x \in \mathbb{R}$,*

(ii) $f(x) > 0$ *for any $x \in \mathbb{R}$,*

(iii) $f(x+y) = f(x)f(y)$ *for any $x, y \in \mathbb{R}$.*

From now on, the notation e^x for $f(x)$ will be used. The previous relations read $\mathrm{e}^x\mathrm{e}^{-x} = 1$, $\mathrm{e}^x > 0$ and $\mathrm{e}^{x+y} = \mathrm{e}^x\mathrm{e}^y$. Since $\big(\mathrm{e}^x\big)' = \mathrm{e}^x > 0$, this function is invertible and its inverse is denoted by ln. More precisely, $\ln : \mathbb{R}_+ \to \mathbb{R}$ satisfies $\mathrm{e}^{\ln(y)} = y$ and $\ln(\mathrm{e}^x) = x$ for any $x \in \mathbb{R}$ and $y \in \mathbb{R}_+$.

Exercise 2 *Prove the following properties of the function* ln *:*

(i) $\ln(y)' = \frac{1}{y}$ *for any $y \in \mathbb{R}_+$,*

(ii) $\ln(yz) = \ln(y) + \ln(z)$ *for any $y, z \in \mathbb{R}_+$,*

(iii) $\ln(y^x) = x\ln(y)$ *for any $y \in \mathbb{R}_+$ and $x \in \mathbb{Q}$.*

Based on the previous exercises, it is natural to set for any $x \in \mathbb{R}$ and $y \in \mathbb{R}_+$

$$y^x := \mathrm{e}^{\ln(y^x)} = \mathrm{e}^{x\ln(y)}$$

Exercise 3 *Let us set $\varepsilon := \mathrm{e}^1 = 2.718...$. Check that $\ln(\varepsilon) = 1$ and that $\varepsilon^x = \mathrm{e}^x$.*

Exercise 4 *Compute the following limits:*

$$a)\ \lim_{x\to 0_+} x\ln(x), \qquad b)\ \lim_{x\to 0_+} x^x, \qquad c)\ \lim_{x\to+\infty} \frac{\ln(x)}{x}, \qquad d)\ \lim_{x\to+\infty} x^{1/x}\ .$$

What can you say for $\lim_{x\to 0_+} x^r\ln(x)$ for any $r > 0$?

Exercise 5 *Compute the following limits:*

$$a)\ \lim_{x\to 0}\frac{\ln(1+x)}{x}, \quad b)\ \lim_{x\to 0}(1+x)^{1/x}, \quad c)\ \lim_{x\to\infty}\big(1+\frac{1}{x}\big)^x, \quad d)\ \lim_{x\to\infty}\big(1+\frac{r}{x}\big)^x \text{ for any } r > 0.$$

Exercise 6 *Compute the derivative of the following functions:*

$$f : \mathbb{R} \ni x \mapsto a^x \in \mathbb{R} \text{ for any } a > 0, \qquad g : \mathbb{R}_+^* \ni x \mapsto x^x \in \mathbb{R}.$$