Nagoya University, G30 program Fall 2016

Mathematics Tutorial Ia (calculus) Instructor : Serge Richard

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Homework 5

Exercise 1 *Show that there are exactly two tangent lines to the graph of the function* $f : \mathbb{R} \ni x \mapsto (x+1)^2 \in \mathbb{R}$ *which pass through the origin. Find the equation of these lines (*$\Leftrightarrow$ *find the two functions whose graphs correspond to these straight lines).*

Exercise 2 *Consider the function* $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = x^2 \sin(1/x)$ *if* $x \neq 0$ *and* $f(0) = 0$.

1. *Show that* f *is continuous at* 0,
2. *Compute the derivative of* f *at* 0,
3. *Compute the derivative of* f *at any* $x \neq 0$,
4. *Show that the derivative of* f *is well-defined but that this derivative is not continuous at* 0.

<u>*Indication:*</u> *One has* $\lim_{y \to 0} \frac{\sin(y)}{y} = 1$, *which will be proved later on.*

Exercise 3 *1) For* $n \in \mathbb{N}^*$ *let* $P_{\frac{1}{n}} : \mathbb{R}_+ \to \mathbb{R}$ *be the function defined by* $P_{\frac{1}{n}}(x) = x^{\frac{1}{n}}$. *Show that the following equality holds:*

$$P'_{\frac{1}{n}}(x) = \frac{1}{n} x^{\frac{1}{n}-1}$$

For the proof you can use the equality

$$(a^n - b^n) = (a-b) \sum_{k=0}^{n-1} a^{n-k-1} b^k$$

for $a = (x+h)^{\frac{1}{n}}$ *and* $b = x^{\frac{1}{n}}$.

2) Deduce that if $P_{\frac{m}{n}} : \mathbb{R}_+ \to \mathbb{R}$ *is defined for* $m, n \in \mathbb{N}^*$ *by* $P_{\frac{m}{n}}(x) = x^{\frac{m}{n}}$, *then*

$$P'_{\frac{m}{n}}(x) = \frac{m}{n} x^{\frac{m}{n}-1} \; .$$

Exercise 4 *Find the equation of the tangent of the curve in* $\mathbb{R}^2$ *defined by the relation*

$$F(x,y) = x^2 - y^2 + 3xy + 12 = 0$$

at the point $(-4, 2)$.

Exercise 5 *By using the indication mentioned above, compute the following limits:*

1. $\lim_{h \to 0} \frac{\cos(h)-1}{h}$,
2. $\lim_{h \to 0} \frac{\cos(h)-1}{h^2}$.