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**Homework 14**

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**Exercise 1** Prove the following statement (the ration test) :

Let  $\{a_j\}_{j=1}^{\infty}$  be a sequence with positive terms only. Assume that there exists  $c \in (0, 1)$  and  $N \in \mathbb{N}$  such that

$$\frac{a_{j+1}}{a_j} \leq c \quad \forall j \geq N.$$

Then the corresponding series  $a_1 + a_2 + a_3 + \dots$  is convergent.

**Exercise 2** Show that the series with generic term  $a_j = \frac{j}{3^j}$  is convergent, in other terms show that

$$\sum_{j=1}^{\infty} \frac{j}{3^j} < \infty.$$

**Exercise 3** For  $f : [1, \infty) \rightarrow \mathbb{R}_+$  decreasing, prove the following statement (the integral test) :

The series  $f(1) + f(2) + f(3) + \dots$  is convergent if and only if  $\lim_{M \rightarrow \infty} \int_1^M f(x) dx$  is convergent, or in a more simple form show that

$$\sum_{j=1}^{\infty} f(j) < \infty \iff \int_1^{\infty} f(x) dx < \infty.$$

**Exercise 4** For any  $\varepsilon > 0$  show that the series with generic term  $a_j = j^{-1-\varepsilon}$  is convergent.

**Exercise 5** Prove the following statement:

Let  $\{a_j\}_{j=1}^{\infty}$  be a sequence of real numbers satisfying

- (i)  $\lim_{j \rightarrow \infty} a_j = 0$ ,
- (ii)  $a_j a_{j+1} \leq 0$  for all  $j \in \mathbb{N}$ ,
- (iii)  $|a_{j+1}| \leq |a_j|$  for all  $j \in \mathbb{N}$ .

Then the corresponding series is convergent.