

ALTERNATING CONVERGENCE

do not assume that the terms a_n converge absolutely if the series

is a_n converges. This is now a simple application of the tests for convergence, which are important, because we have:

sequence, and assume that the

is equal to a_n itself if $a_n \geq 0$. Let $-a_n$ if $a_n \leq 0$. Then both a_n^+ and a_n^- in comparison with $\sum |a_n|$, we see that

$$\sum_{n=1}^{\infty} a_n^-$$

is our theorem.

convergence of a series which may

Theorem 5.2. Let $\sum_{n=1}^{\infty} a_n$ be a series such that

$$\lim_{n \rightarrow \infty} a_n = 0,$$

such that the terms a_n are alternately positive and negative, and such that $|a_{n+1}| \leq |a_n|$ for $n \geq 1$. Then the series is convergent.

Proof. Let us write the series in the form

$$b_1 - c_1 + b_2 - c_2 + b_3 - c_3 + \cdots,$$

with $b_n, c_n \geq 0$. Let

$$s_n = b_1 - c_1 + b_2 - c_2 + \cdots + b_n,$$

$$t_n = b_1 - c_1 + b_2 - c_2 + \cdots + b_n - c_n.$$

Since the absolute values of the terms decrease, it follows that

$$s_1 \geq s_2 \geq s_3 \geq \cdots \quad \text{and} \quad t_1 \leq t_2 \leq t_3 \leq \cdots,$$

i.e. that the s_n are decreasing and the t_n are increasing. Indeed,

$$s_{n+1} = s_n - c_n + b_{n+1} \quad \text{and} \quad 0 \leq b_{n+1} \leq c_n.$$

Thus we subtract more from s_n by c_n than we add afterwards by b_{n+1} . Hence $s_n \geq s_{n+1}$. Furthermore, $s_n \geq t_n$. Hence we may visualize our sequences as follows:

$$s_n \geq s_{n+1} \geq \cdots \geq t_{n+1} \geq t_n.$$

Note that $s_n - t_n = c_n$, and that c_n approaches 0 as n becomes large. If we let L be the greatest lower bound for the sequence $\{s_n\}$, and M be the least upper bound for the sequence $\{t_n\}$, then

$$s_n \geq L \geq M \geq t_n$$

for all n . Since the difference $s_n - t_n$ becomes arbitrarily small, it follows that $L - M$ is arbitrarily small, and hence equal to 0. Thus $L = M$, and this proves that s_n and t_n approach L as a limit, whence our series $\sum a_n$ converges to L .