TOEPLITZ OPERATORS AND THEIR APPLICATIONS

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1. Fourier Analysis and the Hardy Space H^p

1.1. **The Hardy space H^p.** Let $\mathbb{T}$ be the circle group and endow $\mathbb{T}$ with the normalised arc length measure(= Haar measure), denoted by $d\lambda$. We write $L^p(\mathbb{T})$ for $L^p(\mathbb{T}, d\lambda)$. Since $d\lambda(\mathbb{T}) < \infty$, $L^q(\mathbb{T}) \subset L^p(\mathbb{T})$ if $1 \leq p < q$. If $f \in L^1(\mathbb{T})$, then

$$\int f(\lambda)d\lambda = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})dt.$$

For each $n \in \mathbb{Z}$, we define the continuous function ε_n to be

$$\varepsilon_n : \mathbb{T} \to \mathbb{T}, \lambda \mapsto \lambda^n.$$

We denote by Γ and Γ_+ linear spans of the sets $\{\varepsilon_n \mid n \in \mathbb{Z}\}$ and $\{\varepsilon_n \mid n \in \mathbb{N}\}$ respectively. We call the elements in Γ and Γ_+ *trigonometric polynomials* and *analytic trigonometric polynomials*, respectively.

Lemma 1.1. *(1) Γ is a $*$-subalgebra of $C(\mathbb{T})$.*
(2) For $1 \leq p \leq +\infty$, Γ is L^p-norm dense in $L^p(\mathbb{T})$.
(3) $(\varepsilon_n)_{n\in\mathbb{Z}}$ is an orthonormal basis of the Hilbert space $L^2(\mathbb{T})$.

Proof. (1) is clear. By the Stone-Weierstrass theorem, Γ is norm-dense in $C(\mathbb{T})$, and since $C(\mathbb{T})$ is L^p-norm dense in $L^p(\mathbb{T})$, one gets the statement (2). (3) follows immediately from (2). □

Definition 1.2. *If $f \in L^1(\mathbb{T})$ and $n \in \mathbb{Z}$, the n-th Fourier coefficient of f is defined to be*

$$\hat{f}(n) = \int f(\lambda)\overline{\varepsilon_n(\lambda)}d\lambda,$$

and the function

$$\hat{f} : \mathbb{Z} \to \mathbb{C}, n \mapsto \hat{f}(n)$$

is called the Fourier transform of f.

Definition 1.3. *Suppose $1 \le p \le +\infty$. We define the Hardy space H^p by setting,*

$$H^p = \{f \in L^p(\mathbb{T}) \mid \hat{f}(n) = 0 \ (n < 0)\}.$$

H^p is a L^p-norm closed vector subspace of $L^p(\mathbb{T})$. In particular, H^2 endowed with the L^2-scalar product is a Hilbert space with an orthonormal basis Γ_+.

The Hardy spaces have an interpretation of analytic functions on the open unit disk $\mathbb{D} \subset \mathbb{C}$ satisfying certain growth conditions approaching the boundary $\partial\mathbb{D} = \mathbb{T}$. Let us review this for the case $p = 2$. If $f : \mathbb{D} \to \mathbb{C}$ is an analytic function and $0 \le r < 1$, then set $f_r(\lambda) := f(r\lambda)$ $(\lambda \in \mathbb{T})$ and $\|f\|_{H^2(\mathbb{D})} := \sup_{0<r<1} \|f_r\|_{L^2(\mathbb{T})}$. We define *the Hardy space $H^2(\mathbb{D})$ for the unit disk* to be

$$H^2(\mathbb{D}) = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is analytic on } \mathbb{D} \text{ and } \|f\|_{H^2(\mathbb{D})} < +\infty\}.$$

If $z \in \mathbb{D}$, we define $\tau_z : H^1 \to \mathbb{C}$ by setting

$$\tau_z(f) = \int \frac{f(\lambda)}{1 - z\overline{\lambda}} d\lambda \quad (f \in H^1).$$

Since $\frac{1}{1-z\overline{\lambda}} = \sum_{n=0}^{\infty} z^n \overline{\lambda}^n$, one infers that $\tau_z(f) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$. By a direct computation, τ_z is a bounded linear functional on H^1. So, the function

$$\tilde{f} : \mathbb{D} \to \mathbb{C}, \ z \mapsto \tau_z(f),$$

is analytic. Moreover, for each $z \in \mathbb{D}$ and $f,\, g \in H^2$, $\tau_z(fg) = \tau_z(f)\tau_z(g)$.[1]

The relation between H^2 and $H^2(\mathbb{D})$ is explained in the next theorem.

Fact 1.4 (Exercise 3.10 of [1])**.**

1. *$f \in H^2(\mathbb{D})$ if and only if there exists a sequence $(a_n)_{n\in\mathbb{N}} \in \ell^2$ such that $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ $(\lambda \in \mathbb{D})$. Moreover, $||f||_{H^2(\mathbb{D})} = (\sum_{n=0}^{\infty} |a_n|^2)^{\frac{1}{2}}$.*
2. *$H^2(\mathbb{D})$ is a Hilbert space with the scalar product*

$$\langle f, g\rangle_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} a_n \overline{b_n},$$

where $a_n = f^{(n)}(0)/n!$ and $b_n = g^{(n)}(0)/n!$.

3. *The map*

$$H^2 \to H^2(\mathbb{D}), \ f \mapsto \tilde{f},$$

is a unitary operator.

So, we can consider the functions in H^2 as boundary values of functions in $H^2(\mathbb{D})$ by the unitary operator in (3) of Fact 1.4.[2]

Next lemma shows another example of this analytic-type behavior.

Lemma 1.5. *If $f, \bar{f} \in H^1$, then there exists $\alpha \in \mathbb{C}$ such that $f = \alpha$ a.e.*

Proof. Suppose first $f = \bar{f}$ a.e. If $\alpha = \int f(\lambda) d\lambda \in \mathbb{C}$, then

$$\bar{\alpha} = \int \overline{f(\lambda)} d\lambda = \int f(\lambda) d\lambda = \alpha,$$

so $\alpha \in \mathbb{R}$. If $n < 0$, then

$$(\widehat{f - \alpha\varepsilon_0})(n) = \int (f(\lambda) - \alpha)\overline{\varepsilon_n(\lambda)} d\lambda = 0 - \alpha 0 = 0.$$

And also if $n > 0$,

$$(\widehat{f - \alpha\varepsilon_0})(n) = \int (f(\lambda) - \alpha)\overline{\varepsilon_n(\lambda)} d\lambda = \overline{\int (f(\lambda) - \alpha)\overline{\varepsilon_{-n}(\lambda)} d\lambda} = 0,$$

[1] To show this, first consider the case where $f,\, g \in \Gamma_+$. If $f,\, g \in H^2$, then $(\widehat{fg})(n) = 0$ $(n < 0)$ by a direct computation, and therefore $fg \in H^2$ by Hölder's inequality. There exist $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}} \subset \Gamma_+$ such that $f_n \to f$ and $g_n \to g$ $(n \to \infty)$ in the L^2-norm. Using Hölder's inequality again, one gets $f_n g_n \to fg$ $(n \to \infty)$ in the L^1-norm. By the boundedness of τ_z, $\tau_z(fg) = \lim_{n\to\infty} \tau_z(f_n g_n) = \lim_{n\to\infty} \tau_z(f_n)\tau_z(g_n) = \tau_z(f)\tau_z(g)$.

[2] There is a more explicit way to see this boundary behavior using the Poisson kernel P_r. See section 1.5 of [3].

since α is real and $f = \bar{f}$ a.e. Finally,

$$\widehat{(f - \alpha\varepsilon_0)}(0) = \int (f(\lambda) - \alpha) d\lambda = \alpha - \alpha = 0,$$

and then $f = \alpha$ a.e.

Now we suppose only that $f, \bar{f} \in H^1$, then $Re(f)$ and $Im(f)$ are real-valued and in H^1, so by the first paragraph, these functions are constant a.e., and therefore f is constant a.e. □

1.2. **Invariant Subspaces for the Bilateral and Unilateral Shifts.** In this section, we first look at the relation between a multiplication operator and the bilateral shift of the basis (ε_n), and determine the invariant subspaces for it.

If $\varphi \in L^\infty(\mathbb{T})$, the multiplication operator $M_\varphi \in B(L^2(\mathbb{T}))$ with symbol φ is defined by

$$M_\varphi(f) = \varphi f \ (f \in L^2(\mathbb{T})).$$

The map

$$M_* : L^\infty(\mathbb{T}) \to B(L^2(\mathbb{T})), \ \varphi \mapsto M_\varphi$$

is an isometric $*$-homomorphism.

Lemma 1.6. *(1) H^∞ is a closed subalgebra of $L^\infty(\mathbb{T})$.*
(2) Suppose $\varphi \in L^\infty(\mathbb{T})$. Then H^2 is invariant for M_φ if and only if $\varphi \in H^\infty$.

Proof. (1) We only show that if φ, $\psi \in H^\infty$, then $\varphi\psi \in H^\infty$ because other verifications are obvious. If $n \in \mathbb{Z}$, then $\widehat{(\varphi\psi)}(n) = \sum_{m=-\infty}^{\infty} \hat{\varphi}(m)\hat{\psi}(n-m)$. So, if $n < 0$, then $\widehat{(\varphi\psi)}(n) = 0$, and therefore $\varphi\psi \in H^\infty$. (2) follows from (1). □

Notation 1.7. *Set $v := M_{\varepsilon_1}$ and $u := v|_{H^2}$.*

Observe that v is a unitary element in the C^*-algebra $B(L^2(\mathbb{T}))$ and that for each $n \in \mathbb{Z}$, $v(\varepsilon_n) = \varepsilon_{n+1}$. So v is the bilateral shift on the basis $(\varepsilon_n)_{n\in\mathbb{Z}}$ of $L^2(\mathbb{T})$. Similarly, u is the unilateral shift on the basis $(\varepsilon_n)_{n\in\mathbb{N}}$ of H^2.

Theorem 1.8. *If $w \in B(L^2(\mathbb{T}))$, then $wv = vw$ if and only if $w = M_\varphi$ for some $\varphi \in L^\infty(\mathbb{T})$.*

Proof. We only the forward implication because the other one is clear. Suppose $wv = vw$. If $\psi \in \Gamma$, then M_ψ is a linear span of the operators v^n $(n \in \mathbb{Z})$, so M_ψ commutes with w. If ψ is an arbitrary element of $L^\infty(\mathbb{T})$, then there is a sequence $(\psi_n)_{n\in\mathbb{N}} \subset \Gamma$ such that

$$\lim_{n\to\infty} \|\psi - \psi_n\|_{L^2} = 0.$$

Hence, $\lim_{n\to\infty} \|w(\psi) - w(\psi_n)\|_{L^2} = 0$ since w is continuous on $L^2(\mathbb{T})$. And going to subesequence if necessary, we may suppose that $(\psi_n)_{n\in\mathbb{N}}$ converges to ψ a.e. and $(w(\psi_n))_{n\in\mathbb{N}}$ converges to $w(\psi)$ a.e. Now, we set $\varphi = w(\varepsilon_0)$. For each $n \in \mathbb{N}$,

$$w(\psi_n) = wM_{\psi_n}(\varepsilon_0) = M_{\psi_n}w(\varepsilon_0) = \psi_n\varphi \ a.e.$$

Hence, $w(\psi) = \psi\varphi = \varphi\psi$ a.e.

Let $E_n = \{\lambda \in \mathbb{T} \mid |\varphi(\lambda)| > \|w\| + 1/n\}$. Clearly, E_n is a measurable set, and since for each $n \in \mathbb{N}$ one has

$$\begin{aligned}
\|w\|^2\|\chi_{E_n}\|_{L^2}^2 &\geq \|w(\chi_{E_n})\|_{L^2}^2 \\
&= \int |\varphi(\lambda)|^2 \chi_{E_n}(\lambda) d\lambda \\
&\geq \left(\|w\| + \frac{1}{n}\right)^2 \int \chi_{E_n}(\lambda) d\lambda \\
&= \left(\|w\| + \frac{1}{n}\right)^2 \|\chi_{E_n}\|_{L^2}^2,
\end{aligned}$$

one infers that E_n is of measure zero. Hence, $\cup_{n=1}^{\infty} E_n = \{\lambda \in \mathbb{T} \mid |\varphi(\lambda)| > \|w\|\}$ is of measure zero, that is $|\varphi(\lambda)| \leq \|w\|$ a.e. and therefore $\varphi \in L^\infty(\mathbb{T})$. Because $w|_{L^\infty(\mathbb{T})} = M_\varphi$, one deduces that $w = M_\varphi$ on $L^2(\mathbb{T})$ since $L^\infty(\mathbb{T})$ is L^2-norm dense in $L^2(\mathbb{T})$. □

Definition 1.9. *(1) If E is a Borel set of $\mathbb{T}$, then we call range K_E of the projection M_{χ_E} on $L^2(\mathbb{T})$ a Wiener vector subspace of $L^2(\mathbb{T})$.*
(2) If φ is a unitary element in the C^*-algebra $L^\infty(\mathbb{T})$, then the closed vector subspace φH^2 of $L^2(\mathbb{T})$ is called a Beurling vector subspace.

Notice that $v(K_E) = K_E$ (and therefore, K_E reduces v). This is because $M_{\chi_E} v = v M_{\chi_E}$ by Theorem 1.8. Also, if $\varphi \in L^\infty(\mathbb{T})$ is a unitary element, then φH^2 is invariant for v, but $v(\varphi H^2) \neq \varphi H^2$. If $v(\varphi H^2) = \varphi H^2$, then $H^2 = v(H^2) = u(H^2)$, so the unilateral shift is surjective and therefore invertible, which is a contradiction.

Next theorem shows the characterization of the invariant subspaces for the bilateral shift v.

Theorem 1.10. *If K is a closed subspace of $L^2(\mathbb{T})$ invariant for v, then:*
(1) $v(K) = K \iff K$ is a Wiener space,
(2) $v(K) \neq K \iff K$ is a Beurling space.

Proof. By the observation made before the statement, we only have to prove the two forward implications.

(1) Suppose $v(K) = K$, and let p be the orthogonal projection of $L^2(\mathbb{T})$ onto K. Since K reduces v, one gets $pv = vp$, therefore by Theorem 1.8, there exists $\varphi \in L^\infty(\mathbb{T})$ such that $p = M_\varphi$. Since p is a projection, so is φ, and therefore $\varphi = \chi_E$ where E is a measurable set. This shows that K is a Wiener space.

(2) Suppose now $v(K) \neq K$. Then there is a unit vector $\varphi \in K$ such that $\varphi \perp v(K)$. Since $v^n(K) \in v(K)$ for all $n > 0$, it follows

$$0 = \langle v^n(\varphi), \varphi \rangle = \int \varepsilon_n(\lambda) |\varphi(\lambda)|^2 d\lambda.$$

Therefore, for any $n \in \mathbb{Z}\backslash\{0\}$, we have $\int \varepsilon_n(\lambda)|\varphi(\lambda)|^2 d\lambda = 0$, and so $|\varphi|^2 = \alpha$ a.e. for some $\alpha \in \mathbb{R}$ by Lemma 1.5. Since the L^2-norm of φ is 1, then $\alpha = 1$. Thus, φ is a unitary element in $L^\infty(\mathbb{T})$, and clearly, $\varphi H^2 \subset K$. Also, $(\varepsilon_n \varphi)_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$, and $\varepsilon_n \varphi \in K^\perp$ for $n < 0$. So $(\varepsilon_n \varphi)_{n\in\mathbb{N}}$ is an orthonormal basis for K, and therefore $K = \varphi H^2$. Thus K is a Beurling space. □

Now we have two applications of Theorem 1.10.

Theorem 1.11. *The only closed vector subspaces of H^2 reducing for the unilateral shit u are the trivially invariant spaces 0 and H^2.*

Proof. Suppose K is a non-trivial closed vector subspace of H^2 reducing for u. Since $\cap_{n=1}^{\infty} u^n(K) \subseteq \cap_{n=0}^{\infty} u^n(H^2) = 0$ and $K \neq 0$, therefore $v(K) = u(K) \neq K$, so K is a Beurling space by Theorem 1.10. Similarly, $H^2 \ominus K$ is a Beurling space. Hence, there are unitary elements $\varphi, \psi \in L^\infty(\mathbb{T})$ such that,

$$K = \varphi H^2 \quad \text{and} \quad H^2 \ominus K = \psi H^2.$$

For all $n \geq 0$, we have $\varepsilon_n \varphi \in K$ and $\varepsilon_n \psi \in H^2 \ominus K$, so

$$\langle \varepsilon_n \varphi, \psi \rangle = \langle \varphi, \varepsilon_n \psi \rangle = 0$$

Hence, $\varphi\bar{\psi}$ has zero Fourier transform, so $\varphi\bar{\psi} = 0$ a.e.. This is a contradiction because φ and ψ are unitary elements. Thus, the only reducing closed vector subspaces for u are 0 and H^2. □

Theorem 1.12 (F. and M. Riesz)**.** *If $f \in H^2$ does not vanish a.e., then $d\lambda\big(f^{-1}(\{0\})\big) = 0$.*

Proof. Let K be the L^2-norm closed vector subspace of H^2 consisting of all elements $g \in H^2$ such that $g\chi_{f^{-1}(\{0\})} = 0$ a.e. Then K is invariant for u, and therefore v, and so $\cap_{n=1}^{\infty} u^n(K) \subseteq \cap_{n=0}^{\infty} u^n(H^2) = 0$. Hence, if $v(K) = K$, then $K = 0$, and therefore since $f \in K$, $f = 0$ a.e. This contradicts the hypothesis. Hence, $v(K) \neq K$, so by Theorem 1.10, $K = \varphi H^2$ for some unitary element $\varphi \in L^\infty(\mathbb{T})$. Consequently, $\varphi\chi_{f^{-1}(\{0\})} = 0$ a.e., so $\chi_{f^{-1}(\{0\})} = 0$ a.e. Therefore, $d\lambda\big(f^{-1}(\{0\})\big) = 0$. □

2. Toeplitz Operators

2.1. Toeplitz Operators and Toeplitz Matrices.

Let p be the projection of $L^2(\mathbb{T})$ onto H^2.

Definition 2.1. *If $\varphi \in L^\infty(\mathbb{T})$, then the Toeplitz operator T_φ with symbol φ is the operator*

$$T_\varphi : H^2 \to H^2, \ f \mapsto p(\varphi f)$$

(that is, the compression of M_φ to H^2).

Clearly, $\|T_\varphi\| \leq \|\varphi\|_\infty$ and the map

$$T_* : L^\infty(\mathbb{T}) \to B(H^2), \ \varphi \mapsto T_\varphi$$

is linear and preserves adjoint; that is, $T_\varphi^* = T_{\bar{\varphi}}$. Therefore, if $\bar{\varphi} = \varphi$, then T_φ is self-adjoint.

A *Toeplitz matrix* $(a_{ij})_{i,j\in\mathbb{N}}$ is a matrix that is constant along diagonals; that is, $a_{ij} = a_{i+1,j+1}$ for all i,j. Thus a Toeplitz matrix looks like this;

$$\begin{matrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \ddots \\ a_2 & a_1 & a_0 & a_{-1} & \ddots \\ a_3 & a_2 & a_1 & a_0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{matrix}$$

Clearly, a Toeplitz matrix is determined by a two-sided sequence $(a_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$, with $a_{ij} = a_{i-j}$. Since a Toeplitz operator commutes with the unilateral shift u, it is clear that the matrix of a Toeplitz operator T_φ with respect to the basis $(\varepsilon_n)_{n\in\mathbb{N}}$ is a Toeplitz matrix. Moreover, the two-sided sequence corresponding to the matrix of T_φ is $(\hat{\varphi}(n))_{n\in\mathbb{Z}}$.

Remark 2.2. *One can prove that if $a \in B(H^2)$ has a matrix with respect to the basis $(\varepsilon_n)_{n\in\mathbb{N}}$ which is a Toeplitz matrix, then a is a Toeplitz operator.*

The difficulties in this theory come from the observation that the linear map T_* is not multiplicative. Specifically, $T_\varphi T_\psi$ is rarely equal to $T_{\varphi\psi}$. For example, if $\varphi = \varepsilon_1$ and $\psi = \varepsilon_{-1}$, then $T_\varphi = u$ and $T_\psi = u^*$. As $u^*(\varepsilon_0) = 0$, $uu^* \neq 1$, but $T_{\varphi\psi} = T_{\varepsilon_0} = 1$, so $T_\varphi T_\psi \neq T_{\varphi\psi}$.

However, we now have an explicit formula for the matrix of a product of two Toeplitz operators. Let $(a_{i-j})_{i,j\in\mathbb{N}}$ and $(b_{i-j})_{i,j\in\mathbb{N}}$ be the matrices of the Topelitz operators T_φ and T_ψ respectively. If $(c_{i,j})_{i,j\in\mathbb{N}}$ is the matrix of the operator $T_\varphi T_\psi$, then for all i, j,

$$c_{i+1,j+1} = c_{ij} + a_{i+1}b_{-j-1}$$

This equation is called the *product matrix formula* in the paper [2]. The proof is straightforward. This formula play an important role to determine the condition that $T_\varphi T_\psi$ is a Toeplitz operator.

Theorem 2.3. *Let $\varphi, \psi \in L^\infty(\mathbb{T})$. If $\psi \in H^\infty$, then $T_{\varphi\psi} = T_\varphi T_\psi$ and $T_{\bar{\psi}\varphi} = T_{\bar{\psi}} T_\varphi$. Conversely, if $T_\varphi T_\psi$ is a Toeplitz operator, then $\bar{\varphi}$ or $\psi \in H^\infty$, and $T_\varphi T_\psi = T_{\varphi\psi}$*

Proof. If $\psi \in H^\infty$, then clearly $\psi H^2 \subseteq H^2$. If $f \in H^2$, then

$$T_\varphi T_\psi(f) = p(\varphi(p\psi f)) = p(\varphi\psi f) = T_{\varphi\psi}(f),$$

so $T_\varphi T_\psi = T_{\varphi\psi}$. Therefore, $T_{\bar{\varphi}} T_\psi = T_{\bar{\varphi}\psi}$, so by taking adjoints,

$$T_{\bar{\psi}} T_\varphi = T_\psi^* T_{\bar{\varphi}}^* = T_{\bar{\varphi}\psi}^* = (T_{\overline{\varphi}} T_\psi)^* = T_{\bar{\psi}\varphi}.$$

We now suppose conversely that $T_\varphi T_\psi$ is a Toeplitz operator. Then its matrix is a Toeplitz matrix, so by the product matrix formula, $a_{i+1}b_{-j-1} = 0$ for all i, j. It follows that $a_{i+1} = 0$ for each $i \geq 0$ or $b_{-j-1} = 0$ for each $j \geq 0$, which is equivalent to the desired conclusion. □

The following example shows the picture of Theorem 2.3. Let a, b, c, d and $e \in \mathbb{C}$, and set $\psi = a\varepsilon_0 + b\varepsilon_1$ and $\varphi = c\varepsilon_{-1} + d\varepsilon_0 + e\varepsilon_1$. Then, $\psi \in \Gamma_+ \subseteq H^\infty$ and $\varphi \in L^\infty(\mathbb{T})$. So, the matrix Ψ of T_ψ looks like this:

$$\Psi = \begin{matrix} a & 0 & 0 & 0 & \cdots \\ b & a & 0 & 0 & \ddots \\ 0 & b & a & 0 & \ddots \\ 0 & 0 & b & 0 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{matrix}.$$

Similarly, the matrix Φ of T_φ looks like this:

$$\Phi = \begin{matrix} d & c & 0 & 0 & \cdots \\ e & d & c & 0 & \ddots \\ 0 & e & d & c & \ddots \\ 0 & 0 & e & d & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{matrix}.$$

Clearly, these are Toeplitz matrices. Then, the product[3]

$$\Phi\Psi = \begin{matrix} ad+bc & ac & 0 & 0 & \cdots \\ ae+bd & ad+bc & ac & 0 & \ddots \\ be & ae+bd & ad+bc & ac & \ddots \\ 0 & be & ae+bd & ad+bc & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{matrix}$$

is a Toeplitz matrix. Since $\varphi\psi = ac\varepsilon_{-1} + (ad+bc)\varepsilon_0 + (ae+bd)\varepsilon_1 + be\varepsilon_2$, this is the matrix of the Toeplitz operator $T_{\varphi\psi}$.

Corollary 2.4 (Brown-Halmos)**.** *There are no zero divisiors in the set of all Toeplitz operators. Specifically, if $\varphi\psi \in L^\infty(\mathbb{T})$, then*

$$T_\varphi T_\psi = 0 \;\Leftrightarrow\; T_\varphi = 0 \text{ or } T_\psi = 0.$$

Proof. The implication $\Leftarrow$ is trivial, so we prove the converse. Since 0 is a Toeplitz operator, it follows that $\bar{\varphi}$ or $\psi \in H^\infty(\subset H^2)$ and $\varphi\psi = 0$ a.e. by Theorem 2.3. By the F. and M. Riesz theorem, if $\bar{\varphi} \in H^2$, then $\psi = 0$ a.e., and if $\psi \in H^2$, then $\varphi = 0$ a.e. Thus, $T_\varphi = 0$ or $T_\psi = 0$. □

This corollary naturally leads to the following question:

Question 2.5. *Suppose $\varphi_i \in L^\infty(\mathbb{T})$ $(i = 1, 2, \ldots, n)$ and*

$$T_{\varphi_1} T_{\varphi_2} \cdots T_{\varphi_n} = 0.$$

Then, is it necessary that there exists an index i such that $T_{\varphi_i} = 0$?

This problem, which is a natural generalization of Corollary 2.4, has been solved completely by Alexandru Aleman and Dragan Vukotić [4] in 2009, by showing that the question above has an affirmative answer for all n.

[3]In general, matrix multiplication of two infinite matrices is not defined. However, in this case, since every rows and columns of Φ and Ψ have only finitely many nonzero entries, the product is well-defined. Strictly speaking, we have to consider corresponding two-sided sequences, and then compute it by the product matrix formula.

2.2. **Elementary Spectral Theory of Toeplitz Operators.** Now, we study the elementary spectral theory of Toeplitz operators. First, we apply the F. and M. Riesz theorem to Toeplitz operators.

Lemma 2.6. *If $\varphi \in H^\infty$ and φ is not a scalar a.e., then T_φ has no eigenvalues.*

Proof. Suppose that $f \in H^2$ and $\lambda \in \mathbb{C}$ and
$$(T_\varphi - \lambda)(f) = 0.$$
Then, $(\varphi-\lambda)f = 0$ a.e. Since $\varphi-\lambda \in H^2$ and is not the zero element, the set $\{\zeta \in \mathbb{C} \mid (\varphi-\lambda)(\zeta) = 0\}$ is of measure 0 by the F. and M. Riesz theorem. Thus $f = 0$ a.e. □

Recall that if A is a unital C*-algebra with the unit 1, and $a \in A$, then the *spectrum* $\sigma(a)$ of a in A is defined to be
$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda 1 - a \text{ is not invertible in } A\ \}.$$
And the *spectral radius* of a is
$$r(a) = \sup_{\lambda\in\sigma(a)} |\lambda|.$$

Theorem 2.7 (Hartman-Wintner)**.** *Let $\varphi \in L^\infty(\mathbb{T})$ and let $\sigma(\varphi)$ denote the spectrum of φ in $L^\infty(\mathbb{T})$. Then*
$$\sigma(\varphi) \subseteq \sigma(T_\varphi) \text{ and } r(T_\varphi) = \|T_\varphi\| = \|\varphi\|_\infty.$$

Proof. To show that $\sigma(\varphi) \subseteq \sigma(T_\varphi)$, it suffices to show that if T_φ is invertible in $B(H^2)$, then φ is invertible in $L^\infty(\mathbb{T})$. Indeed, this reduction follows from the equality $T_\varphi - \lambda = T_{\varphi-\lambda}$ if $\lambda \in \mathbb{C}$. So, we now suppose T_φ is invertible and set $m := \|T_\varphi^{-1}\|$. For all $f \in H^2$,
$$\|f\| = \|T_\varphi^{-1}T_\varphi(f)\| \leq m\|T_\varphi(f)\|.$$
One then infers that for any $n \in \mathbb{Z}$
$$\|M_\varphi(\varepsilon_n f)\| = \|\varphi\varepsilon_n f\| = \|\varphi\| \geq \|T_\varphi(f)\| \geq \frac{\|f\|}{m} = \frac{\|\varepsilon_n f\|}{m}.$$
However, the functions $\varepsilon_n f$ are L^2-norm dense in $L^2(\mathbb{T})$, since Γ is L^2-norm dense in $L^2(\mathbb{T})$. Hence, for all $g \in L^2(\mathbb{T})$ we have $\|M_\varphi(g)\| \geq \|g\|/m$, and so $M_\varphi^* M_\varphi \geq m^{-2} > 0$. It follows that $M_\varphi^* M_\varphi$ is invertible, and by the normality of M_φ, M_φ is invertible. Since the map $M_* : L^\infty(\mathbb{T}) \to B(L^2(\mathbb{T}))$ is an isometric $*$-homomorphism (and then injective), φ is invertible in $L^\infty(\mathbb{T})$

Now, suppose that φ is an arbitrary element of $L^\infty(\mathbb{T})$. Then, since $\sigma(\varphi) \subseteq \sigma(T_\varphi)$, we get $\|T_\varphi\| \leq \|\varphi\| = r(\varphi) \leq r(T_\varphi) \leq \|T_\varphi\|$, so we have $\|T_\varphi\| = r(T_\varphi) = \|\varphi\|$. □

Notation 2.8 (Von Neumann-Schatten product)**.** *If H is a Hilbert space and $x, y \in H$, then we define an operator $x \otimes y^c$ on H by setting $(x \otimes y^c)(z) = \langle z, y\rangle_H x$ $(z \in H)$. We call $x \otimes y^c$ the von Neumann-schatten product of x and y.*

One can easily see that $x \otimes y^c$ is a bounded operator of rank 1 if x and y are not trivial. Other basic properties of von Neumann-Schatten products are in the section 2.4 of [1].

Notation 2.9. *If H is a Hilbert space, we denote by $K(H)$ and $F(H)$ the set of all the compact operators and the set of all the finite-rank operators on H respectively.*

Theorem 2.10. *If $\varphi \in L^\infty(\mathbb{T})$, then T_φ is compact if and only if $\varphi = 0$.*

Proof. Observe first that
$$u^{*n}(\varepsilon_m) = \begin{cases} \varepsilon_{m-n} & \text{if } m \geq n \\ 0 & \text{if } m < n. \end{cases}$$
Therefore, if $f \in H^2$, then
$$\|u^{*n}(f)\|^2 = \|\sum_{m=n}^{\infty} \langle f, \varepsilon_m\rangle \varepsilon_{m-n}\|^2 = \sum_{m=n}^{\infty} |\langle f, \varepsilon_m\rangle|^2.$$

So, $u^{*n}(f) \to 0$ $(n \to \infty)$ in H^2. If $v \in F(H^2)$, then there exists $f_1, \dots, f_n$ and $g_1, \dots, g_n$ in H^2 such that $v = \sum_{j=1}^n f_j \otimes g_j^c$, and therefore for each positive integer m, we have $u^{*m}v = \sum_{j=1}^n u^{*m}(f_j) \otimes g_j^c$. So, $\lim_{m\to\infty} \|u^{*m}v\| = 0$, and since $F(H^2)$ is dense in $K(H^2)$, $\lim_{m\to\infty} \|u^{*m}w\| = 0$ for all $w \in K(H^2)$. Observe that $u^*T_\varphi u = T_{\varepsilon_{-1}} T_\varphi T_{\varepsilon_1} = T_{\varepsilon_{-1}\varphi\varepsilon_1} = T_\varphi$ by Theorem 2.3, so if T_φ is compact, then since

$$\|T_\varphi\| = \|u^{*m} T_\varphi u^{*m}\| \leq \|u^{*m} T_\varphi\| \text{ and } \lim_{m\to\infty} \|u^{*m} T_\varphi\| = 0,$$

we have $T_\varphi = 0$. By the Hartman-Winter's theorem (Theorem 2.7), $\varphi = 0$.

□

The following lemma is one of the keys to determine the commutator ideal of the Toeplitz algebra.

Lemma 2.11. *If $\varphi \in C(\mathbb{T})$ and $\psi \in L^\infty(\mathbb{T})$, then $T_\varphi T_\psi - T_{\varphi\psi}$ and $T_\psi T_\varphi - T_{\psi\varphi}$ are compact operators.*

Proof. Since $K(H^2)$ is self-adjoint, we only have to show that $T_\psi T_\varphi - T_{\psi\varphi} \in K(H^2)$. Since Γ is dense in $C(\mathbb{T})$ and the map T_* is linear, we may suppose that $\varphi = \varepsilon_n$ for some $n \in \mathbb{Z}$. If $n \geq 0$, by Theorem 2.3. $T_\psi T_{\varepsilon_n} - T_{\psi\varepsilon_n} = 0 \in K(H^2)$.

Now, we show that $T_\psi T_{\varepsilon_{-k}} - T_{\psi\varepsilon_{-k}}$ is compact for all $k \geq 1$, by induction on k. If $f \in H^2$, then $T_\psi T_{\varepsilon-1}(f) = p(\psi p(\varepsilon_{-1} f) = p(\psi(\varepsilon_{-1} f - \langle f, \varepsilon_0 \rangle \varepsilon_{-1})) = T_{\psi\varepsilon_{-1}} f - \langle f, \varepsilon_0 \rangle p(\psi\varepsilon_{-1})$. Hence, $T_\psi T_{\varepsilon_{-1}} - T_{\psi\varepsilon_{-1}}$ is an operator of rank not greater than 1.

Suppose that we have shown that $T_\psi T_{\varepsilon_{-k}} - T_{\psi\varepsilon_{-k}}$ is compact for $\psi \in L^\infty(\mathbb{T})$ and some $k \geq 1$. Then

$$T_\psi T_{\varepsilon_{-k-1}} - T_{\psi\varepsilon_{-k-1}} = (T_\psi T_{\varepsilon_{-k}} - T_{\psi\varepsilon_{-k}}) T_{\varepsilon_{-1}} + T_{\psi\varepsilon_{-k}} T_{\varepsilon_{-1}} - T_{(\psi\varepsilon_{-k})\varepsilon_{-1}}$$

is compact. This proves the result.

□

3. The Toeplitz Algebra

3.1. **The Toeplitz Algebra.** Let $\mathbb{A}$ denote the C*-algebra generated by all Toeplitz operators T_φ with symbol $\varphi \in C(\mathbb{T})$, and call $\mathbb{A}$ the *Toeplitz algebra*.

If A is a C*-algebra, then its *commutator ideal* is the closed ideal generated by the commutators $[a, b] = ab - ba$ $(a, b \in A)$. Clearly, the commutator ideal is the smallest closed ideal I in A such that A/I is abelian. First, we determine the commutator ideal of the Toeplitz algebra $\mathbb{A}$.

Theorem 3.1. *The commutator ideal of $\mathbb{A}$ is $K(H^2)$.*

Proof. observe first that if K is a closed vector subspace of H^2 invariant for $\mathbb{A}$, then K is reducing for u, and therefore by Theorem 1.11 $K = 0$ or H^2. Thus, $\mathbb{A}$ is an irreducible subalgebra of $B(H^2)$.

Now, set $p = [u^*, u] = 1 - uu^*$. Obviously, p is of rank 1 and belongs to the commutator ideal I of $\mathbb{A}$. Since finite-rank operators are compact, $p \in \mathbb{A} \cap K(H^2)$. By the Theorem 2.4.9 in [1] (and the irreducibility in one of its assumption), it follows that $K(H^2) \subseteq \mathbb{A}$. Since $\mathbb{A}/K(H^2)$ is generated by the elements

$$T_\varphi + K(H^2) \quad (\varphi \in C(\mathbb{T})),$$

which are commuting and normal by Lemma 2.10, $\mathbb{A}/K(H^2)$ is abelian. Because $K(H^2)$ is simple, $I = K(H^2)$. □

Theorem 3.2. *The map*

$$\Psi : C(\mathbb{T}) \to \mathbb{A}/K(H^2), \ \varphi \mapsto T_\varphi + K(H^2)$$

is a $*$-isomorphism.

Proof. If φ, $\psi \in C(\mathbb{T})$, by Lemma 2.10

$$\Psi(\varphi\psi) = T_{\varphi\psi} + K(H^2) = T_\varphi T_\psi + K(H^2) = \Psi(\varphi)\Psi(\psi).$$

So, Ψ is multiplicative, and since the map T_* is linear and preserves adjoint, so is Ψ. Thus, Ψ is a $*$-homomorphism. Clearly, the elements $T_\varphi + K(H^2)$ ($\varphi \in C(\mathbb{T})$) generate $\mathbb{A}/K(H^2)$, and therefore Ψ is surjective. If $\Psi(\varphi) = T_\varphi + K(H^2) = K(H^2)$, then $T_\varphi \in K(H^2)$, and therefore by Theorem 2.9, $\varphi = 0$. Thus, Ψ is injective. □

Now, we recall indispensable items in the operator theorist's tool-kit; the index and the essential spectrum. Let X, Y be Banach spaces. $w \in B(X, Y)$ is *Fredholm* if $\dim(\ker w)$ and $\mathrm{codim}_Y(w(X))$ are finite. We define the *index* of w to be

$$\mathrm{ind}(w) = \dim(\ker w) - \mathrm{codim}_Y(w(X)).$$

We will frequently use the following fact (Theorem 1.4.8 of [1]) to calculate Fredholm indices.

Fact 3.3. *Let X, Y, Z be Banach spaces, and let $w_1 : X \to Y$ and $w_2 : Y \to Z$ be Fredholm operators. Then $w_2 w_1$ is Fredholm and*

$$\mathrm{ind}(w_2 w_1) = \mathrm{ind}(w_2) + \mathrm{ind}(w_1).$$

Since all the operators between finite-dimensional spaces are Fredholm, we suppose X is an infinite-dimensional Banach space. We have a useful characterization of Fredholm operators.

Fact 3.4 (Atkinson characterization)**.** *If $w \in B(X)$, then*

$$w \textit{ is Fredholm} \Leftrightarrow w + K(X) \textit{ is invertible in } B(X)/K(X).$$

The *essential spectrum* $\sigma_e(w)$ of $w \in B(X)$ is defined to be

$$\sigma_e(w) = \{\lambda \in \mathbb{C} \mid w - \lambda \text{ is not Fredholm}\}.$$

If $\pi : B(X) \to B(X)/K(X)$ is the quotient map, then it is clear that $\sigma_e(w) = \sigma(\pi(w))$ by the Atkinson characterization.

Corollary 3.5. *If $\varphi \in C(\mathbb{T})$, then T_φ is Fredholm if and only if φ vanishes nowhere.*

Proof. By the Atkinson characterization,

$$\begin{aligned} T_\varphi \textit{ is Fredholm} &\Leftrightarrow T_\varphi + K(H^2) \textit{ is invertible in } B(H^2)/K(H^2) \\ &\Leftrightarrow T_\varphi + K(H^2) \textit{ is invertible in } \mathbb{A}(H^2)/K(H^2). \end{aligned}$$

Therefore, by Theorem 3.2, T_φ is Fredholm if and only if φ is invertible in $C(\mathbb{T})$. □

Corollary 3.6. *If $\varphi \in C(\mathbb{T})$, then $\sigma_e(T_\varphi) = \varphi(\mathbb{T})$.*

Proof. By the Atkinson characterizaton and Theorem 3.2, $\sigma_e(T_\varphi) = \sigma(\pi(T_\varphi)) = \sigma(\varphi) = \varphi(\mathbb{T})$. □

Hence, a Toeplitz operator with continuous symbol has connected essential spectrum. In particular, $\sigma_e(u) = \sigma(T_{\varepsilon_1}) = \mathbb{T}$.

3.2. **The Baby Index Theorem.** Now, we calculate the Fredholm index of Toeplitz operators T_{ε_n}. Since $\ker u = 0$ and $u(H^2)$ is the closed linear span of the set $\{\varepsilon_n \mid n \geq 1\}$, so $\mathrm{ind}(T_{\varepsilon_1}) = \mathrm{ind}(u) = 0 - 1 = -1$. So, the unilateral shift u has index -1. If $n \in \mathbb{N}$, then $\mathrm{ind}(T_{\varepsilon_n}) = \mathrm{ind}(u^n) = n\,\mathrm{ind}(u) = -n$, where Fact 3.3 has been used in the second equality. By, the equality $T_{\varepsilon_{-n}} T_{\varepsilon_n} = T_1$ one also infers that $\mathrm{ind}(T_{\varepsilon_{-n}}) = n = -(-n)$.

We will generalize this result, which is the simplest case of the Atiyah-Singer index theorem on odd-dimensional manifolds.

Theorem 3.7 (The baby index theorem)**.** *If φ is an invertible element in $C(\mathbb{T})$, then*

$$\mathrm{ind}(T_\varphi) = -\,\mathrm{wn}(\varphi),$$

where $\mathrm{wn}(\varphi)$ *is the winding number of* φ *with respect to the origin.*

The winding number of the continuous function $\varphi : \mathbb{T} \to \mathbb{C}\backslash\{0\}$ is the unique integer n given by next lemma.

Lemma 3.8. *If φ is an invertible function in $C(\mathbb{T})$. then there exists a unique integer $n \in \mathbb{Z}$ such that $\varphi = \varepsilon_n e^{\psi}$ for some $\psi \in C(\mathbb{T})$.*

Proof. We only show the uniqueness of the integer n (for the existence of n and ψ; see the proof of Lemma 3.5.14 of [1]). We need only show that if $\varepsilon_n = e^{\psi}$ for some $\psi \in C(\mathbb{T})$, then $n = 0$. The map

$$\alpha : [0,1] \to \mathbb{Z},\ t \mapsto \mathrm{ind}(T_{e^{t\psi}})$$

is continuous and has discrete range and connected domain, so it is necessarily constant. Hence, $-n = \mathrm{ind}(T_{e^{\psi}}) = \alpha(1) = \alpha(0) = \mathrm{ind}(T_1) = \mathrm{ind}(1) = 0$. □

Proof of Theorem 3.6. Suppose that $\varphi \in C(\mathbb{T})$ is an invertible element with $\mathrm{wn}(\varphi) = n$. Now, $\varphi = \varepsilon_n e^{\psi}$ for some $\psi \in C(\mathbb{T})$, and $T_\varphi = T_{\varepsilon_n} T_{e^{\psi}}$ if $n \geq 0$ and $T_\phi = T_{\epsilon_n} T_{e^{\psi}}$ if $n < 0$, by Theorem 2.3. Hence,

$$\mathrm{ind}(T_\varphi) = \mathrm{ind}(\varepsilon_n) + \mathrm{ind}(e^{\psi}) = -n$$

since $T_{e^{\psi}}$ is of index zero by the proof of Lemma 3.8. □

Since the winding number of φ is unchanged as long as the homotopy class is fixed, so is the "analytic invariant", the Fredholm index of a Toeplitz operator T_φ.

Corollary 3.9. *Let $\varphi \in C(\mathbb{T})$ is an invertible element. Then the following conditions are equivalent;*

(1) T_φ is invertible,
(2) $\mathrm{ind}(T_\varphi) = 0$,
(3) $\varphi = e^{\psi}$ for some $\psi \in C(\mathbb{T})$.

Proof. We need only to show the implication (3)⇒(1) since the others are obvious from the preceding result.

First, if $\rho = \sum_{|n|\leq N} \lambda_n \varepsilon_n \in \Gamma$, write $\rho' = \sum_{n=0}^{N} \lambda_n \varepsilon_n$ and $\rho'' = \sum_{n=1}^{N} \lambda_{-n} \varepsilon_{-n}$. Then, $\rho = \rho' + \rho''$ and $\rho', \overline{\rho''} \in H^\infty$. By (1) of Lemma 1.6., it follows that $e^{\rho'}, e^{\overline{\rho''}} \in H^\infty$. By using Theorem 2.3, one gets $T_{e^{\rho'}}$ and $T_{e^{\overline{\rho''}}}$ are invertible with inverses $T_{e^{-\rho'}}$ and $T_{e^{-\overline{\rho''}}}$ respectively.

Now, if $\psi \in C(\mathbb{T})$ is an arbitrary element, then by the density of Γ in $C(\mathbb{T})$, we may choose a trigonometric polynomial ρ as above such that $\|1 - e^{\psi-\rho}\| < 1$. Then $T_{e^{\psi}} = T_{e^{\rho''} e^{\psi-\rho} e^{\rho'}} = T_{e^{\rho''}} T_{e^{\psi-\rho}} T_{e^{\rho'}}$. Since $\|1 - T_{e^{\psi-\rho}}\| = \|T_{1-e^{\psi-\rho}}\| = \|1 - e^{\psi-\rho}\| < 1$, the operator $T_{e^{\psi-\rho}}$ is invertible. Hence, $T_{e^{\psi}}$ is a product of invertible operators and is therefore invertible. □

We have an application of the baby index theorem to the spectral theory of Toeplitz operator.

Theorem 3.10. *The spectrum of a Toeplitz operator with continuous symbol is connected.*

Proof. If $\varphi \in C(\mathbb{T})$, then by the baby index theorem and Corollary 3.8,

$$\sigma(T_\varphi) = \varphi(\mathbb{T}) \cup \{\lambda \in \mathbb{C} \mid \mathrm{wn}(\varphi - \lambda) \neq 0\}$$

Therefore, $\sigma(T_\varphi)$ is a compact set consisting of the connected compact curve $\varphi(\mathbb{T})$ and some of its holes, and therefore $\sigma(T_\varphi)$ is connected. □

As Question 2.5, a natural question arises from this theorem.

Question 3.11. *Is the spectrum of a Toeplitz operator necessarily connected?*

This question was posed by Halmos, and then Harold Widom [6] answered by proving the following theorem.

Theorem 3.12 (H. Widom)**.** *Every Toeplitz operator has a connected spectrum.*

3.3. **Coburn's Theorem.** In this last section, we first prove an important structure theorem for isometries in a C*-algebra, which is an application of the Toeplitz algebra. Then, we consider the universal property of the Toeplitz algebra $\mathbb{A}$ (Coburn's theorem).

Remark 3.13. *Since $C(\mathbb{T})$ is a C^*-algebra generated by ε_1, $\mathbb{A}$ is genrated by the unilateral shift $u = T_{\varepsilon_1}$.*

Theorem 3.14 (Wold-von Neumann)**.** *If W is a isometry on a Hilbrt space H, then W is a unitary, or a direct sum of copies of the unilateral shift, or a direct sum of a unitary and copies of the unilateral shift.*

Remark 3.15. *Let α be a cardinal number. We denote by $u^{(\alpha)}$ the direct sum of α copies of the unilateral shift u. Theorem 3.13 says that an isometry W is of the form*

$$W = u^{(\alpha)} \oplus U,$$

where U is a unitary (possibly vacuous).

Proof. We may suppose that W is neither a unitary nor a sum of copies of the unilateral shift. Set

$$K = \bigcap_{n=0}^{\infty} W^n(H).$$

Then, $W(K) = K$, and therefore K reduces W. We set $W = w \oplus w'$ where w and w' are compressions of W to K and $K^{\perp}$ respectively. Since w is an isometry, the equation $W(K) = K$ implies w is surjective, therefore w is a unitary. So, we need only to prove that w' is a direct sum of copies of the unilateral shift u.

Now, set

$$L = (W(H))^{\perp}.$$

Then, $W^n(L) \subseteq W(H) = L^{\perp}$ for all $n > 0$, so if $m, n \in \mathbb{N}$ and $m \neq n$, then $W^m(L) \perp W^n(L)$. We claim that

$$\bigoplus_{n=0}^{\infty} W^n(L) = K^{\perp}.$$

Clearly, $W^n(L) \perp W^n(L^{\perp}) (= W^{n+1}(H))$. Since $K \subseteq W^{n+1}(H)$, one infers that $W^n(L) \subseteq K^{\perp}$, and therefore $\oplus_{n=0}^{\infty} W^n(L) \subseteq K^{\perp}$. To show $\oplus_{n=0}^{\infty} W^n(L) \supseteq K^{\perp}$, it suffices to show that if $x \in H$ such that $x \perp W^n(L)$ for all $n \in \mathbb{N}$, then $x \in K$. Obviously, this is true for $n = 0$. If $x \in W^n(H)$, then there exists $y \in H$ such that $x = W^n(y)$. Since $W^n(y) \perp W^n(L)$, then $y \in L^{\perp} = W(H)$, and therefore $x \in W^{n+1}(H)$.

Let E be an orthonormal basis for L. Since W is a isometry, $\cup_{n=0}^{\infty} W^n(E)$ is an orthonormal basis for $K^{\perp}$. For each $e \in E$, let L_e be the Hilbert subspace of $K^{\perp}$ having $(W^n(e))_{n=0}^{\infty}$ as orthonormal basis. Then,

$$K^{\perp} = \bigoplus_{e \in E} L_e,$$

each L_e is invariant for W, the complession W_e of W to L_e is the unilateral shift, and

$$w' = \bigoplus_{e \in E} W_e.$$

□

Before going to Coburn's theorem, we recall a basic tool in the operator theory ([1] section 2.1).

Fact 3.16 (the functional calculus)**.** *Let B be a unital C^*-algebra, a be a normal element in B and $z : \sigma(a) \to \mathbb{C}$ be the inclusion map. Then there exists a unique unital isometric $*$-homomorphism*

$$\Theta_a : C(\sigma(a)) \to B$$

such that $\Theta_a(z) = a$ and the image $\Theta_a(C(\sigma(a)))$ is the C^*-algebra generated by the unit of B and a.

The map Θ_a is called the *functional calculus* at a. Note that if $f \in C(\sigma(a))$, then we write $\Theta_a(f) = f(a)$.

Lemma 3.17. *If w is a unitary in a unital C*-algebra B. Then there is a unique unital $*$-homomorphism $\varphi : C(\mathbb{T}) \to B$ such that $\varphi(z) = w$.*

Proof. Observe that $\sigma(w) \subseteq \mathbb{T}$ and that the map

$$i : C(\mathbb{T}) \to C(\sigma(w)),\ f \mapsto f|_{\sigma(w)}$$

is a unital $*$-homomorphism. One get φ by setting $\varphi = \Theta_w \circ i$. Since z generates $C(\mathbb{T})$, φ is unique. $\square$

Theorem 3.18 (Coburn)**.** *Suppose that w is an isometry in a unital C*-algebra B. Then there exists a unique unital $*$-homomorphism*

$$\Phi : \mathbb{A} \to B$$

such that $\Phi(u) = w$. Moreover, if $ww^* \neq 1$ (that is w is not unitary), then φ is isometric.

Proof. The uniqueness of Φ is clear by Remark 3.13. By the GNS-construction, we reduce to the case where B is a unital C*-subalgebra of $B(H)$ containing id_H for some Hilbert space H. By the Wold-von Neumann decomposition, we can write

$$\begin{cases} H = \oplus_{\lambda\in\Lambda} H_\lambda, \\ w = \oplus_{\lambda\in\Lambda} w_\lambda, \end{cases}$$

where H_λ are Hilbert spaces and each $w_\lambda \in B(H_\lambda)$ is a unitary or a unilateral shift.

If w_λ is a unitary, then by Theorem 3.2 and Lemma 3.17 and by composing all the following maps

$$\mathbb{A} \xrightarrow{\ \pi\ } \mathbb{A}/K(H^2) \xrightarrow{\ \Psi^{-1}\ } C(\mathbb{T}) \xrightarrow{\ \varphi\ } B(H_\lambda),$$

where π is the quotient map, one gets a unital $*$-homomorphism $\Phi_\lambda : \mathbb{A} \to B(H_\lambda)$ such that $\Phi_\lambda(u) = w_\lambda$.

If w_λ is a unilateral shift, then there exists a unitary $U_\lambda : H^2 \to H_\lambda$ such that $w_\lambda = U_\lambda u U_\lambda^*$. Hence, the map

$$\Phi_\lambda : \mathbb{A} \to B(H_\lambda),\ a \mapsto U_\lambda a U_\lambda^*,$$

is an isometric unital $*$-homomorphism such that $\Phi_\lambda(u) = w_\lambda$.

Now, by preceding paragraphs, we get the family of representations $(H_\lambda, \Phi_\lambda)_{\lambda\in\Lambda}$ of the C*-algebra $\mathbb{A}$. Let (H, Φ) be the direct sum of $(H_\lambda, \Phi_\lambda)_{\lambda\in\Lambda}$. Then $\Phi : \mathbb{A} \to B(H)$ is a unital $*$-homomorphism such that $\Phi(u) = \oplus_\lambda w_\lambda = w$. Moreover, since $\Phi(u) \in B$ and u generates $\mathbb{A}$, one deduces that $\Phi(\mathbb{A}) \subseteq B$.

Finally, we suppose that $ww^* \neq 1$. Then some w_{λ_0} is a unilateral shift. Hence, the representation $(H_{\lambda_0}, \Phi_{\lambda_0})$ is faithful, so is (H, Φ). Therefore, using the fact that an injective $*$-homomorphism between C*-algebras is necessarily isometric, Φ is isometric.

$\square$

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