TOEPLITZ OPERATORS AND THEIR APPLICATIONS

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1. Fourier Analysis and the Hardy Space $H^p$

1.1. The Hardy space $H^p$. Let $\mathbb{T}$ be the circle group and endow $\mathbb{T}$ with the normalised arc length measure (Haar measure), denoted by $d\lambda$. We write $L^p(\mathbb{T})$ for $L^p(\mathbb{T}, d\lambda)$. Since $d\lambda(\mathbb{T}) < \infty$, $L^q(\mathbb{T}) \subset L^p(\mathbb{T})$ if $1 \leq p < q$. If $f \in L^1(\mathbb{T})$, then

$$\int f(\lambda)d\lambda = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it})dt.$$ 

For each $n \in \mathbb{Z}$, we define the continuous function $\varepsilon_n$ to be $\varepsilon_n : \mathbb{T} \to \mathbb{T}, \lambda \mapsto \lambda^n$. We denote by $\Gamma$ and $\Gamma_+$ linear spans of the sets $\{\varepsilon_n \mid n \in \mathbb{Z}\}$ and $\{\varepsilon_n \mid n \in \mathbb{N}\}$ respectively. We call the elements in $\Gamma$ and $\Gamma_+$ trigonometric polynomials and analytic trigonometric polynomials, respectively.

Lemma 1.1. (1) $\Gamma$ is a $*$-subalgebra of $C(\mathbb{T})$.
(2) For $1 \leq p \leq +\infty$, $\Gamma$ is $L^p$-norm dense in $L^p(\mathbb{T})$.
(3) $(\varepsilon_n)_{n \in \mathbb{Z}}$ is an orthonormal basis of the Hilbert space $L^2(\mathbb{T})$.

Proof. (1) is clear. By the Stone-Weierstrass theorem, $\Gamma$ is norm-dense in $C(\mathbb{T})$, and since $C(\mathbb{T})$ is $L^p$-norm dense in $L^p(\mathbb{T})$, one gets the statement (2). (3) follows immediately from (2). □

Definition 1.2. If $f \in L^1(\mathbb{T})$ and $n \in \mathbb{Z}$, the $n$-th Fourier coefficient of $f$ is defined to be

$$\hat{f}(n) = \int f(\lambda)\varepsilon_n(\lambda)d\lambda,$$

and the function

$$\hat{f} : \mathbb{Z} \to \mathbb{C}, n \mapsto \hat{f}(n)$$

is called the Fourier transform of $f$. 

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Definition 1.3. Suppose $1 \leq p \leq +\infty$. We define the Hardy space $H^p$ by setting,

$$H^p = \{ f \in L^p(T) \mid \hat{f}(n) = 0 (n < 0) \}.$$ 

$H^p$ is a $L^p$-norm closed vector subspace of $L^p(T)$. In particular, $H^2$ endowed with the $L^2$-scalar product is a Hilbert space with an orthonormal basis $\Gamma^+$.

The Hardy spaces have an interpretation of analytic functions on the open unit disk $D \subset \mathbb{C}$ satisfying certain growth conditions approaching the boundary $\partial D = \Gamma$. Let us review this for the case $p = 2$. If $f : D \to \mathbb{C}$ is an analytic function and $0 \leq r < 1$, then set $f_r(\lambda) := f(r\lambda)$ ($\lambda \in T$) and $\|f\|_{H^2(D)} := \sup_{0 < r < 1} \|f_r\|_{L^2(T)}$. We define the Hardy space $H^2(D)$ for the unit disk to be

$$H^2(D) = \{ f : D \to \mathbb{C} \mid f \text{ is analytic on } D \text{ and } \|f\|_{H^2(D)} < +\infty \}.$$ 

If $z \in D$, we define $\tau_z : H^1 \to \mathbb{C}$ by setting

$$\tau_z(f) = \int \frac{f(\lambda)}{1 - z\lambda} d\lambda \quad (f \in H^1).$$ 

Since $\frac{1}{1 - z\lambda} = \sum_{n=0}^{\infty} z^n \lambda^n$, one infers that $\tau_z(f) = \sum_{n=0}^{\infty} f(n)z^n$. By a direct computation, $\tau_z$ is a bounded linear functional on $H^1$. So, the function

$$\tilde{f} : D \to \mathbb{C}, \quad z \mapsto \tau_z(f),$$

is analytic. Moreover, for each $z \in D$ and $f, g \in H^2$, $\tau_z(fg) = \tau_z(f)\tau_z(g)$.\(^1\)

The relation between $H^2$ and $H^2(D)$ is explained in the next theorem.

Fact 1.4 (Exercise 3.10 of [1]). (1) $f \in H^2(D)$ if and only if there exists a sequence $(a_n)_{n \in \mathbb{N}} \in \ell^2$ such that $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ ($\lambda \in D$). Moreover, $\|f\|_{H^2(D)} = (\sum_{n=0}^{\infty} |a_n|^2)^{\frac{1}{2}}$.

(2) $H^2(D)$ is a Hilbert space with the scalar product

$$\langle f, g \rangle_{H^2(D)} = \sum_{n=0}^{\infty} a_n b_n,$$

where $a_n = f^{(n)}(0)/n!$ and $b_n = g^{(n)}(0)/n!$.

(3) The map

$$H^2 \to H^2(D), \quad f \mapsto \tilde{f},$$

is a unitary operator.

So, we can define the functions in $H^2$ as boundary values of functions in $H^2(D)$ by the unitary operator in (3) of Fact 1.4.\(^2\)

Next lemma shows another example of this analytic-type behavior.

Lemma 1.5. If $f, \tilde{f} \in H^1$, then there exists $\alpha \in \mathbb{C}$ such that $f = \alpha \ a.e.$

Proof. Suppose first $f = \tilde{f}$ a.e. If $\alpha = \int f(\lambda)d\lambda \in \mathbb{C}$, then

$$\alpha = \int \tilde{f}(\lambda)d\lambda = \int f(\lambda)d\lambda = \alpha,$$

so $\alpha \in \mathbb{R}$. If $n < 0$, then

$$(f - \alpha \varepsilon_0)(n) = \int (f(\lambda) - \alpha)\varepsilon_n(\lambda)d\lambda = 0 - \alpha 0 = 0.$$ 

And also if $n > 0$,

$$(f - \alpha \varepsilon_0)(n) = \int (f(\lambda) - \alpha)\varepsilon_n(\lambda)d\lambda = \int (f(\lambda) - \alpha)\varepsilon_{-n}(\lambda)d\lambda = 0,$$

\(^1\)To show this, first consider the case where $f, g \in \Gamma_+$. If $f, g \in H^2$, then $(\tilde{f}g)(n) = 0 (n < 0)$ by a direct computation, and therefore $fg \in H^2$ by Hölder’s inequality. There exist $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}} \subset \Gamma_+$ such that $f_n \to f$ and $g_n \to g (n \to \infty)$ in the $L^2$-norm. Using Hölder’s inequality again, one gets $f_ng_n \to fg (n \to \infty)$ in the $L^1$-norm. By the boundedness of $\tau_z$, $\tau_z(fg) = \lim_{n \to \infty} \tau_z(f_ng_n) = \lim_{n \to \infty} \tau_z(f_n)\tau_z(g_n) = \tau_z(f)\tau_z(g)$.

\(^2\)There is a more explicit way to see this boundary behavior using the Poisson kernel $P_r$. See section 1.5 of [3].
since $\alpha$ is real and $f = \bar{f}$ a.e. Finally,

$$(f - \alpha \varepsilon_0)(0) = \int (f(\lambda) - \alpha)\,d\lambda = \alpha - \alpha = 0,$$

and then $f = \alpha$ a.e.

Now we suppose only that $f, \bar{f} \in H^1$, then $Re(f)$ and $Im(f)$ are real-valued and in $H^1$, so by the first paragraph, these functions are constant a.e., and therefore $f$ is constant a.e. $\square$

1.2. Invariant Subspaces for the Bilateral and Unilateral Shifts. In this section, we first look at the relation between a multiplication operator and the bilateral shift of the basis $(\varepsilon_n)$, and determine the invariant subspaces for it.

If $\varphi \in L^\infty(\mathbb{T})$, the multiplication operator $M_\varphi \in B(L^2(\mathbb{T}))$ with symbol $\varphi$ is defined by

$$M_\varphi(f) = \varphi f \ (f \in L^2(\mathbb{T})).$$

The map

$$M_* : L^\infty(\mathbb{T}) \to B(L^2(\mathbb{T})), \ \varphi \mapsto M_\varphi$$

is an isometric $*$-homomorphism.

**Lemma 1.6.** (1) $H^\infty$ is a closed subalgebra of $L^\infty(\mathbb{T})$.

(2) Suppose $\varphi \in L^\infty(\mathbb{T})$. Then $H^2$ is invariant for $M_\varphi$ if and only if $\varphi \in H^\infty$.

**Proof.** (1) We only show that if $\varphi, \psi \in H^\infty$, then $\varphi \psi \in H^\infty$ because other verifications are obvious. If $n \in \mathbb{Z}$, then $(\varphi \psi)(n) = \sum_{m=-\infty}^{\infty} \hat{\varphi}(m)\hat{\psi}(n-m)$. So, if $n < 0$, then $(\varphi \psi)(n) = 0$, and therefore $\varphi \psi \in H^\infty$. (2) follows from (1). $\square$

**Notation 1.7.** Set $v := M_{\varepsilon_1}$ and $u := v|_{H^2}$.

Observe that $v$ is a unitary element in the $C^*$-algebra $B(L^2(\mathbb{T}))$ and that for each $n \in \mathbb{Z}$, $v(\varepsilon_n) = \varepsilon_{n+1}$. So $v$ is the bilateral shift on the basis $(\varepsilon_n)_{n \in \mathbb{Z}}$ of $L^2(\mathbb{T})$. Similarly, $u$ is the unilateral shift on the basis $(\varepsilon_n)_{n \in \mathbb{N}}$ of $H^2$.

**Theorem 1.8.** If $w \in B(L^2(\mathbb{T}))$, then $wv = vw$ if and only if $w = M_\varphi$ for some $\varphi \in L^\infty(\mathbb{T})$.

**Proof.** We only the forward implication because the other one is clear. Suppose $wv = vw$. If $\psi \in \Gamma$, then $M_\psi$ is a linear span of the operators $v^n \ (n \in \mathbb{Z})$, so $M_0$ commutes with $w$. If $\psi$ is an arbitrary element of $L^\infty(\mathbb{T})$, then there is a sequence $(\psi_n)_{n \in \mathbb{N}} \subset \Gamma$ such that

$$\lim_{n \to \infty} \|\psi - \psi_n\|_{L^2} = 0.$$ 

Hence, $\lim_{n \to \infty} \|w(\psi) - w(\psi_n)\|_{L^2} = 0$ since $w$ is continuous on $L^2(\mathbb{T})$. And going to subsequence if necessary, we may suppose that $(\psi_n)_{n \in \mathbb{N}}$ converges to $\psi$ a.e. and $(w(\psi_n))_{n \in \mathbb{N}}$ converges to $w(\psi)$ a.e. Now, we set $\varphi = w(\varepsilon_0)$. For each $n \in \mathbb{N}$,

$$w(\psi_n) = wM_{\varepsilon_n}(\varepsilon_0) = M_{\varepsilon_n}w(\varepsilon_0) = \psi_n \varphi \ a.e.$$ 

Hence, $w(\psi) = \psi \varphi = \varphi \psi \ a.e.$

Let $E_n = \{ \lambda \in \mathbb{T} \ | \ |\varphi(\lambda)| > \|w\| + 1/n \}$. Clearly, $E_n$ is a measurable set, and since for each $n \in \mathbb{N}$ one has

$$\|w\|^2\|\chi_{E_n}\|^2_{L^2} \geq \|w(\chi_{E_n})\|^2_{L^2}$$

$$= \int |\varphi(\lambda)|^2 \chi_{E_n}(\lambda)\,d\lambda$$

$$\geq \left(\|w\| + \frac{1}{n}\right)^2 \int \chi_{E_n}(\lambda)\,d\lambda$$

$$= \left(\|w\| + \frac{1}{n}\right)^2 \|\chi_{E_n}\|^2_{L^2},$$
one infers that \( E_n \) is of measure zero. Hence, \( \bigcup_{n=1}^{\infty} E_n = \{ \lambda \in \mathbb{T} \mid |\varphi(\lambda)| > \|w\| \} \) is of measure zero, that is \( |\varphi(\lambda)| \leq \|w\| \) a.e. and therefore \( \varphi \in L^\infty(\mathbb{T}) \). Because \( w|_{L^\infty(\mathbb{T})} = M_\varphi \), one deduces that \( w = M_\varphi \) on \( L^2(\mathbb{T}) \) since \( L^\infty(\mathbb{T}) \) is \( L^2 \)-norm dense in \( L^2(\mathbb{T}) \). \( \square \)

**Definition 1.9.**

1. If \( E \) is a Borel set of \( \mathbb{T} \), then we call range \( K_E \) of the projection \( M_{\chi_E} \) on \( L^2(\mathbb{T}) \) a Wiener vector subspace of \( L^2(\mathbb{T}) \).

2. If \( \varphi \) is a unitary element in the \( C^* \)-algebra \( L^\infty(\mathbb{T}) \), then the closed vector subspace \( \varphi H^2 \) of \( L^2(\mathbb{T}) \) is called a Beurling vector subspace.

Notice that \( v(K_E) = K_E \) (and therefore, \( K_E \) reduces \( v \)). This is because \( M_{\chi_E} v = v M_{\chi_E} \) by Theorem 1.8. Also, if \( \varphi \in L^\infty(\mathbb{T}) \) is a unitary element, then \( \varphi H^2 \) is invariant for \( v \), but \( v(\varphi H^2) \neq \varphi H^2 \). If \( v(\varphi H^2) = \varphi H^2 \), then \( H^2 = v(H^2) = u(H^2) \), so the unilateral shift is surjective and therefore invertible, which is a contradiction.

Next theorem shows the characterization of the invariant subspaces for the bilateral shift \( v \).

**Theorem 1.10.** If \( K \) is a closed subspace of \( L^2(\mathbb{T}) \) invariant for \( v \), then:

1. \( v(K) = K \iff K \) is a Wiener space,
2. \( v(K) \neq K \iff K \) is a Beurling space.

**Proof.** By the observation made before the statement, we only have to prove the two forward implications.

1. Suppose \( v(K) = K \), and let \( p \) be the orthogonal projection of \( L^2(\mathbb{T}) \) onto \( K \). Since \( K \) reduces \( v \), one gets \( pv = vp \), therefore by Theorem 1.8, there exists \( \varphi \in L^\infty(\mathbb{T}) \) such that \( p = M_\varphi \). Since \( p \) is a projection, so is \( \varphi \), and therefore \( \varphi = \chi_E \) where \( E \) is a measurable set. This shows that \( K \) is a Wiener space.

2. Suppose now \( v(K) \neq K \). Then there is a unit vector \( \varphi \in K \) such that \( \varphi \perp v(K) \). Since \( v^n(K) \in v(K) \) for all \( n > 0 \), it follows

\[
0 = \langle v^n(\varphi), \varphi \rangle = \int \varepsilon_n(\lambda)|\varphi(\lambda)|^2d\lambda.
\]

Therefore, for any \( n \in \mathbb{Z} \setminus \{0\} \), we have \( \int \varepsilon_n(\lambda)|\varphi(\lambda)|^2d\lambda = 0 \), and so \( |\varphi|^2 = \alpha \) a.e. for some \( \alpha \in \mathbb{R} \) by Lemma 1.5. Since the \( L^2 \)-norm of \( \varphi \) is 1, then \( \alpha = 1 \). Thus, \( \varphi \) is a unitary element in \( L^\infty(\mathbb{T}) \), and clearly, \( \varphi H^2 \subset K \). Also, \( (\varepsilon_n\varphi)_{n \in \mathbb{Z}} \) is an orthonormal basis for \( L^2(\mathbb{T}) \), and \( \varepsilon_n\varphi \in K \perp K \) for \( n < 0 \). So \( (\varepsilon_n\varphi)_{n \in \mathbb{Z}} \) is an orthonormal basis for \( K \), and therefore \( K = \varphi H^2 \). Thus \( K \) is a Beurling space. \( \square \)

Now we have two applications of Theorem 1.10.

**Theorem 1.11.** The only closed vector subspaces of \( H^2 \) reducing for the unilateral shift \( u \) are the trivially invariant spaces \( 0 \) and \( H^2 \).

**Proof.** Suppose \( K \) is a non-trivial closed vector subspace of \( H^2 \) reducing for \( u \). Since \( \cap_{n=1}^{\infty} u^n(K) \subset \cap_{n=0}^{\infty} u^n(H^2) = 0 \) and \( K \neq 0 \), therefore \( v(K) = u(K) \neq K \), so \( K \) is a Beurling space by Theorem 1.10. Similarly, \( H^2 \ominus K \) is a Beurling space. Hence, there are unitary elements \( \varphi, \psi \in L^\infty(\mathbb{T}) \) such that

\[
K = \varphi H^2 \quad \text{and} \quad H^2 \ominus K = \psi H^2.
\]

For all \( n \geq 0 \), we have \( \varepsilon_n\varphi \in K \) and \( \varepsilon_n\psi \in H^2 \ominus K \), so

\[
\langle \varepsilon_n\varphi, \psi \rangle = \langle \varphi, \varepsilon_n\psi \rangle = 0
\]

Hence, \( \varphi \psi \) has zero Fourier transform, so \( \varphi \psi = 0 \) a.e.. This is a contradiction because \( \varphi \) and \( \psi \) are unitary elements. Thus, the only reducing closed vector subspaces for \( u \) are \( 0 \) and \( H^2 \). \( \square \)

**Theorem 1.12** (F. and M. Riesz). If \( f \in H^2 \) does not vanish a.e., then \( d\lambda(f^{-1}(\{0\}))) = 0. \)
Proof. Let $K$ be the $L^2$-norm closed vector subspace of $H^2$ consisting of all elements $g \in H^2$ such that $g\chi_{f^{-1}(\{0\})} = 0$ a.e. Then $K$ is invariant for $u$, and therefore $v$, and so $\cap_{n=1}^{\infty} u^n(K) \subseteq \cap_{n=0}^{\infty} u^n(H^2) = 0$. Hence, if $v(K) = K$, then $K = 0$, and therefore since $f \in K$, $f = 0$ a.e. This contradicts the hypothesis. Hence, $v(K) \neq K$, so by Theorem 1.10, $K = \varphi H^2$ for some unitary element $\varphi \in L^\infty(\mathbb{T})$. Consequently, $\varphi \chi_{f^{-1}(\{0\})} = 0$ a.e., so $\chi_{f^{-1}(\{0\})} = 0$ a.e. Therefore, $d\lambda(f^{-1}(\{0\})) = 0$. 

\section{Toeplitz Operators}

\subsection{Toeplitz Operators and Toeplitz Matrices.}

\begin{definition}
If $\varphi \in L^\infty(\mathbb{T})$, then the Toeplitz operator $T_\varphi$ with symbol $\varphi$ is the operator

$$T_\varphi : H^2 \rightarrow H^2, \ f \mapsto p(\varphi f)$$

(that is, the compression of $M_\varphi$ to $H^2$).

Clearly, $\|T_\varphi\| \leq \|\varphi\|_\infty$ and the map

$$T_\varphi : L^\infty(\mathbb{T}) \rightarrow B(H^2), \ \varphi \mapsto T_\varphi$$

is linear and preserves adjoint; that is, $T_\varphi^* = T_\varphi$. Therefore, if $\tilde{\varphi} = \varphi$, then $T_\varphi$ is self-adjoint.

A Toeplitz matrix $(a_{ij})_{i,j \in \mathbb{N}}$ is a matrix that is constant along diagonals; that is, $a_{ij} = a_{i+1,j+1}$ for all $i,j$. Thus a Toeplitz matrix looks like this;

\[
\begin{array}{cccccc}
  a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\
  a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\
  a_2 & a_1 & a_0 & a_{-1} & \cdots \\
  a_3 & a_2 & a_1 & a_0 & \cdots \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
\end{array}
\]

Clearly, a Toeplitz matrix is determined by a two-sided sequence $(a_n)_{n \in \mathbb{Z}} \subset \mathbb{C}$, with $a_{ij} = a_{i-j}$. Since a Toeplitz operator commutes with the unilateral shift $u$, it is clear that the matrix of a Toeplitz operator $T_\varphi$ with respect to the basis $(\varepsilon_n)_{n \in \mathbb{N}}$ is a Toeplitz matrix. Moreover, the two-sided sequence corresponding to the matrix of $T_\varphi$ is $(\tilde{\varphi}(n))_{n \in \mathbb{Z}}$.

\begin{remark}
One can prove that if $a \in B(H^2)$ has a matrix with respect to the basis $(\varepsilon_n)_{n \in \mathbb{N}}$ which is a Toeplitz matrix, then $a$ is a Toeplitz operator.
\end{remark}

The difficulties in this theory come from the observation that the linear map $T_\varphi$ is not multiplicative. Specifically, $T_\varphi T_\psi$ is rarely equal to $T_{\varphi \psi}$. For example, if $\varphi = \varepsilon_1$ and $\psi = \varepsilon_{-1}$, then $T_\varphi^* \psi = u$ and $T_\psi^* \varphi = u^*$. As $u^*(\varepsilon_0) = 0$, $u u^* \neq 1$, but $T_{\varphi \psi}^* = T_{\varphi^* \psi^*}$, so $T_{\varphi \psi} T_{\varphi^* \psi^*} \neq T_{\varphi \psi}\psi^* \varphi$.

However, we now have an explicit formula for the matrix of a product of two Toeplitz operators.

Let $(a_{i-j})_{i,j \in \mathbb{N}}$ and $(b_{i-j})_{i,j \in \mathbb{N}}$ be the matrices of the Toeplitz operators $T_\varphi$ and $T_\psi$ respectively. If $(c_{i,j})_{i,j \in \mathbb{N}}$ is the matrix of the operator $T_\varphi T_\psi$, then for all $i,j$,

$$c_{i+1,j+1} = c_{ij} + a_{i+1} b_{j+1}$$

This equation is called the product matrix formula in the paper [2]. The proof is straightforward.

This formula play an important role to determine the condition that $T_\varphi T_\psi$ is a Toeplitz operator.

\begin{theorem}
Let $\varphi, \psi \in L^\infty(\mathbb{T})$. If $\psi \in \mathcal{H}_\infty$, then $T_\varphi T_\psi = T_{\varphi \psi} T_\psi$ and $T_\psi T_\varphi = T_\varphi T_{\varphi \psi}$. Conversely, if $T_\varphi T_\psi$ is a Toeplitz operator, then $\varphi$ or $\psi \in H^\infty$, and $T_\varphi T_\psi = T_{\varphi \psi}$.
\end{theorem}

\begin{proof}
If $\psi \in \mathcal{H}_\infty$, then clearly $\psi H^2 \subseteq \mathcal{H}_\infty$. If $f \in H^2$, then

$$T_\varphi T_\psi (f) = p(\varphi (\psi f)) = p(\varphi \psi f) = T_{\varphi \psi}(f),$$

so $T_\varphi T_\psi = T_{\varphi \psi}$. Therefore, $T_\varphi T_\psi = T_{\varphi \psi}$, so by taking adjoints,

$$T_\psi^* T_\varphi^* = T_{\varphi \psi}^* = (T_{\varphi \psi})^* = T_{\psi^* \varphi^*}.$$

\end{proof}
We now suppose conversely that \( T_\varphi T_\psi \) is a Toeplitz operator. Then its matrix is a Toeplitz matrix, so by the product matrix formula, \( a_{i+1}b_{j-1} = 0 \) for all \( i, j \). It follows that \( a_{i+1} = 0 \) for each \( i \geq 0 \) or \( b_{j-1} = 0 \) for each \( j \geq 0 \), which is equivalent to the desired conclusion. \( \square \)

The following example shows the picture of Theorem 2.3. Let \( a, b, c, d \) and \( e \in \mathbb{C} \), and set \( \psi = a\varepsilon_0 + b\varepsilon_1 \) and \( \varphi = c\varepsilon_{-1} + d\varepsilon_0 + e\varepsilon_1 \). Then, \( \psi \in \Gamma_+ \subseteq H^\infty \) and \( \varphi \in L^\infty(T) \). So, the matrix \( \Psi \) of \( T_\psi \) looks like this:

\[
\begin{bmatrix}
  a & 0 & 0 & 0 & \cdots \\
  b & a & 0 & 0 & \cdots \\
  0 & b & a & 0 & \cdots \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]

Similarly, the matrix \( \Phi \) of \( T_\varphi \) looks like this:

\[
\begin{bmatrix}
  d & c & 0 & 0 & \cdots \\
  e & d & c & 0 & \cdots \\
  0 & e & d & c & \cdots \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]

Clearly, these are Toeplitz matrices. Then, the product\(^3\)

\[
\begin{bmatrix}
  ad + bc & ac & 0 & 0 & \cdots \\
  ae + bd & ad + bc & ac & 0 & \cdots \\
  0 & bc & 0 & 0 & \cdots \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]

is a Toeplitz matrix. Since \( \varphi \psi = ac\varepsilon_{-1} + (ad + bc)\varepsilon_0 + (ae + bd)\varepsilon_1 + be\varepsilon_2 \), this is the matrix of the Toeplitz operator \( T_{\varphi \psi} \).

**Corollary 2.4** (Brown-Halmos). There are no zero divisors in the set of all Toeplitz operators. Specifically, if \( \varphi \psi \in L^\infty(T) \), then

\[
T_\varphi T_\psi = 0 \iff T_\varphi = 0 \text{ or } T_\psi = 0.
\]

**Proof.** The implication \( \Rightarrow \) is trivial, so we prove the converse. Since 0 is a Toeplitz operator, it follows that \( \varphi \) or \( \psi \in H^\infty(\subseteq H^2) \) and \( \varphi \psi = 0 \) a.e. by Theorem 2.3. By the F. and M. Riesz theorem, if \( \varphi \in H^2 \), then \( \psi = 0 \) a.e., and if \( \psi \in H^2 \), then \( \varphi = 0 \) a.e. Thus, \( T_\varphi = 0 \) or \( T_\psi = 0 \). \( \square \)

This corollary naturally leads to the following question:

**Question 2.5.** Suppose \( \varphi_i \in L^\infty(T) \) \((i = 1, 2, \ldots, n)\) and

\[
T_{\varphi_1} T_{\varphi_2} \cdots T_{\varphi_n} = 0.
\]

Then, is it necessary that there exists an index \( i \) such that \( T_{\varphi_i} = 0 \)?

This problem, which is a natural generalization of Corollary 2.4, has been solved completely by Alexandru Aleman and Dragan Vukotić [4] in 2009, by showing that the question above has an affirmative answer for all \( n \).

\(^3\)In general, matrix multiplication of two infinite matrices is not defined. However, in this case, since every rows and columns of \( \Phi \) and \( \Psi \) have only finitely many nonzero entries, the product is well-defined. Strictly speaking, we have to consider corresponding two-sided sequences, and then compute it by the product matrix formula.
2.2. Elementary Spectral Theory of Toeplitz Operators. Now, we study the elementary spectral theory of Toeplitz operators. First, we apply the F. and M. Riesz theorem to Toeplitz operators.

**Lemma 2.6.** If $\varphi \in H^\infty$ and $\varphi$ is not a scalar a.e., then $T_\varphi$ has no eigenvalues.

**Proof.** Suppose that $f \in H^2$ and $\lambda \in \mathbb{C}$ and 

$$(T_\varphi - \lambda)(f) = 0.$$ 

Then, $(\varphi - \lambda)f = 0$ a.e. Since $\varphi - \lambda \in H^2$ and is not the zero element, the set \{ $\zeta \in \mathbb{C} \mid (\varphi - \lambda)(\zeta) = 0$\} is of measure 0 by the F. and M. Riesz theorem. Thus $f = 0$ a.e. □

Recall that if $A$ is a unital $C^*$-algebra with the unit 1, and $a \in A$, then the spectrum $\sigma(a)$ of $a$ in $A$ is defined to be 

$$\sigma(a) = \{ \lambda \in \mathbb{C} \mid \lambda a - a \text{ is not invertible in } A \}.$$ 

And the spectral radius of $a$ is 

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$ 

**Theorem 2.7** (Hartman-Wintner). Let $\varphi \in L^\infty(\mathbb{T})$ and let $\sigma(\varphi)$ denote the spectrum of $\varphi$ in $L^\infty(\mathbb{T})$. Then 

$$\sigma(\varphi) \subseteq \sigma(T_\varphi) \text{ and } r(T_\varphi) = \|T_\varphi\| = \|\varphi\|_\infty.$$ 

**Proof.** To show that $\sigma(\varphi) \subseteq \sigma(T_\varphi)$, it suffices to show that if $T_\varphi$ is invertible in $B(H^2)$, then $\varphi$ is invertible in $L^\infty(\mathbb{T})$. Indeed, this reduction follows from the equality $T_\varphi - \lambda = T_{\varphi - \lambda}$ if $\lambda \in \mathbb{C}$. So, we now suppose $T_\varphi$ is invertible and set $m := \|T_\varphi^{-1}\|$. For all $f \in H^2$, 

$$\|f\| = \|T_\varphi^{-1}T_\varphi(f)\| \leq m\|T_\varphi(f)\|.$$ 

One then infers that for any $n \in \mathbb{Z}$ 

$$\|M_\varphi(\varepsilon_n f)\| = \|\varphi \varepsilon_n f\| = \|\varphi\| \geq \|T_\varphi(f)\| \geq \frac{\|f\|}{m} = \frac{\|\varepsilon_n f\|}{m}.$$ 

However, the functions $\varepsilon_n f$ are $L^2$-norm dense in $L^2(\mathbb{T})$, since $\Gamma$ is $L^2$-norm dense in $L^2(\mathbb{T})$. Hence, for all $g \in L^2(\mathbb{T})$ we have $\|M_\varphi(g)\| \geq \|g\|/m$, and so $M_\varphi^* M_\varphi \geq m^{-2} > 0$. It follows that $M_\varphi^* M_\varphi$ is invertible, and by the normality of $M_\varphi$, $M_\varphi$ is invertible. Since the map $M_\varphi : L^\infty(\mathbb{T}) \to B(L^2(\mathbb{T}))$ is an isometric $*$-homomorphism (and then injective), $\varphi$ is invertible in $L^\infty(\mathbb{T})$.

Now, suppose that $\varphi$ is an arbitrary element of $L^\infty(\mathbb{T})$. Then, since $\sigma(\varphi) \subseteq \sigma(T_\varphi)$, we get $\|T_\varphi\| \leq \|\varphi\| = r(\varphi) \leq r(T_\varphi) \leq \|T_\varphi\|$, so we have $\|T_\varphi\| = r(T_\varphi) = \|\varphi\|$. □

**Notation 2.8** (Von Neumann-Schatten product). If $H$ is a Hilbert space and $x, y \in H$, then we define an operator $x \otimes y^*$ on $H$ by setting $(x \otimes y^*)(z) = \langle z, y \rangle_H x$ ($z \in H$). We call $x \otimes y^*$ the von Neumann-schatten product of $x$ and $y$.

One can easily see that $x \otimes y^*$ is a bounded operator of rank 1 if $x$ and $y$ are not trivial. Other basic properties of von Neumann-Schatten products are in the section 2.4 of [1].

**Notation 2.9.** If $H$ is a Hilbert space, we denote by $K(H)$ and $F(H)$ the set of all the compact operators and the set of all the finite-rank operators on $H$ respectively.

**Theorem 2.10.** If $\varphi \in L^\infty(\mathbb{T})$, then $T_\varphi$ is compact if and only if $\varphi = 0$.

**Proof.** Observe first that 

$$u^*(\varepsilon_m) = \begin{cases} \varepsilon_{m-n} & \text{if } m \geq n \\ 0 & \text{if } m < n. \end{cases}$$ 

Therefore, if $f \in H^2$, then 

$$\|u^*(f)\|^2 = \sum_{m=n}^{\infty} \langle f, \varepsilon_m \rangle^2 \varepsilon_{m-n}^2 = \sum_{m=n}^{\infty} |\langle f, \varepsilon_m \rangle|^2.$$
Theorem 2.3. \( C \) dense in \( \mathbb{T} \).

Proof. Since \( \mathbb{T} \) is dense in \( \mathbb{C} \) and \( \mathbb{C} \) is linear, we may suppose that \( \psi \in \mathbb{C} \) for some \( n \in \mathbb{Z} \). If \( n \geq 0 \), by Theorem 2.3, \( T_\psi = T_\varphi \) is compact for \( \varphi \in \mathbb{C} \) and some \( \varphi \). Then \( T_\varphi T_\psi = T_\varphi T_\psi = T_\varphi \psi \) are compact operators.

The following lemma is one of the keys to determine the commutator ideal of the Toeplitz algebra.

Lemma 2.11. If \( \varphi \in C(\mathbb{T}) \) and \( \psi \in L^\infty(\mathbb{T}) \), then \( T_\varphi T_\psi - T_\varphi \psi \) and \( T_\psi T_\varphi - T_\psi \varphi \) are compact operators.

Proof. Since \( K(H^2) \) is self-adjoint, we only have to show that \( T_\varphi T_\psi - T_\varphi \psi \in K(H^2) \). Since \( \Gamma \) is dense in \( C(\mathbb{T}) \) and the map \( T_\varphi \) is linear, we may suppose that \( \varphi = \varepsilon_n \) for some \( n \in \mathbb{Z} \). If \( n \geq 0 \), by Theorem 2.3, \( T_\varphi T_\psi - T_\varphi \psi = 0 \in K(H^2) \).

Now, we show that \( T_\varphi T_\varepsilon - T_\varphi \varepsilon_k \) is compact for all \( k \geq 1 \), by induction on \( k \). If \( f \in H^2 \), then \( T_\varphi T_\varepsilon(f) = \langle \varepsilon(f) \rangle = \langle \varepsilon(f) \rangle = \langle \varepsilon(f) \rangle = T_\varphi \varepsilon(f) - (f, \varepsilon(f)) \psi \varepsilon \). Hence, \( T_\varphi T_\varepsilon = \psi \varphi \) is an operator of rank not greater than 1.

Suppose that we have shown that \( T_\varphi T_\varepsilon_k - T_\varphi \varepsilon_k \) is compact for \( \varepsilon \in L^\infty(\mathbb{T}) \) and some \( k \geq 1 \). Then

\[
T_\varphi T_\varepsilon_k - T_\varphi \varepsilon_k = (T_\varphi T_\varepsilon_k - T_\varphi \varepsilon_k) T_\varepsilon + T_\varphi \varepsilon_k T_\varepsilon - T_\varphi \varepsilon_k \varepsilon
\]

is compact. This proves the result.

3. The Toeplitz Algebra

3.1. The Toeplitz Algebra. Let \( \mathbb{A} \) denote the \( C^* \)-algebra generated by all Toeplitz operators \( T_\varphi \) with symbol \( \varphi \in C(\mathbb{T}) \), and call \( \mathbb{A} \) the Toeplitz algebra.

If \( \mathbb{A} \) is a \( C^* \)-algebra, then its commutator ideal is the closed ideal generated by the commutators \([a, b] = ab - ba \协会 \mathbb{A} \). Clearly, the commutator ideal is the smallest closed ideal \( I \) in \( \mathbb{A} \) such that \( A/I \) is abelian. First, we determine the commutator ideal of the Toeplitz algebra \( \mathbb{A} \).

Theorem 3.1. The commutator ideal of \( \mathbb{A} \) is \( K(H^2) \).

Proof. observe first that if \( \mathbb{A} \) is a closed vector subspace of \( H^2 \) invariant for \( \mathbb{A} \), then \( \mathbb{A} \) is reducing for \( u \), and therefore by Theorem 1.11 \( K = 0 \) or \( H^2 \). Thus, \( \mathbb{A} \) is an irreducible subalgebra of \( B(H^2) \).

Now, set \( p = [u, u^*] = 1 - uu^* \). Obviously, \( p \) is of rank 1 and belongs to the commutator ideal \( I \) of \( \mathbb{A} \). Since finite-rank operators are compact, \( p \in \mathbb{A} \cap K(H^2) \). By Theorem 2.4.9 in [1] (and the irreducibility in one of its assumption), it follows that \( K(H^2) \subseteq \mathbb{A} \). Since \( \mathbb{A}/K(H^2) \) is generated by the elements

\[
T_\varphi + K(H^2) \quad (\varphi \in C(\mathbb{T})),
\]

which are commuting and normal by Lemma 2.10, \( \mathbb{A}/K(H^2) \) is abelian. Because \( K(H^2) \) is simple, \( I = K(H^2) \).

Theorem 3.2. The map

\[
\Psi : C(\mathbb{T}) \to \mathbb{A}/K(H^2), \quad \varphi \mapsto T_\varphi + K(H^2)
\]

is a \( * \)-isomorphism.
Proof. If \( \varphi, \psi \in C(\mathbb{T}) \), by Lemma 2.10
\[
\Psi(\varphi \psi) = T_{\varphi \psi} + K(H^2) = T_\varphi T_\psi + K(H^2) = \Psi(\varphi)\Psi(\psi).
\]
So, \( \Psi \) is multiplicative, and since the map \( T \) is linear and preserves adjoint, so is \( \Psi \). Thus, \( \Psi \) is a \( * \)-homomorphism. Clearly, the elements \( T_\varphi + K(H^2) \) \((\varphi \in C(\mathbb{T})) \) generate \( \mathcal{A}/K(H^2) \), and therefore \( \Psi \) is surjective. If \( \Psi(\varphi) = T_\varphi + K(H^2) = K(H^2) \), then \( T_\varphi \in K(H^2) \), and therefore by Theorem 2.9, \( \varphi = 0 \). Thus, \( \Psi \) is injective. \( \square \)

Now, we recall indispensable items in the operator theorist’s tool-kit; the index and the essential spectrum. Let \( X, Y \) be Banach spaces. \( w \in B(X,Y) \) is Fredholm if \( \dim(\ker w) \) and \( \text{codim}_Y(w(X)) \) are finite. We define the index of \( w \) to be
\[
\text{ind}(w) = \dim(\ker w) - \text{codim}_Y(w(X)).
\]

We will frequently use the following fact (Theorem 1.4.8 of [1]) to calculate Fredholm indices.

**Fact 3.3.** Let \( X, Y, Z \) be Banach spaces, and let \( w_1 : X \to Y \) and \( w_2 : Y \to Z \) be Fredholm operators. Then \( w_2 w_1 \) is Fredholm and
\[
\text{ind}(w_2 w_1) = \text{ind}(w_2) + \text{ind}(w_1).
\]

Since all the operators between finite-dimensional spaces are Fredholm, we suppose \( X \) is an infinite-dimensional Banach space. We have a useful characterization of Fredholm operators.

**Fact 3.4** (Atkinson characterization). If \( w \in B(X) \), then
\( w \) is Fredholm \iff \( w + K(X) \) is invertible in \( B(X)/K(X) \).

The essential spectrum \( \sigma_e(w) \) of \( w \in B(X) \) is defined to be
\[
\sigma_e(w) = \{ \lambda \in \mathbb{C} \mid w - \lambda \text{ is not Fredholm} \}.
\]

If \( \pi : B(X) \to B(X)/K(X) \) is the quotient map, then it is clear that \( \sigma_e(w) = \sigma(w) \) by the Atkinson characterization.

**Corollary 3.5.** If \( \varphi \in C(\mathbb{T}) \), then \( T_\varphi \) is Fredholm if and only if \( \varphi \) vanishes nowhere.

**Proof.** By the Atkinson characterization,
\( T_\varphi \) is Fredholm \iff \( T_\varphi + K(H^2) \) is invertible in \( B(H^2)/K(H^2) \)
\iff \( T_\varphi + K(H^2) \) is invertible in \( \mathcal{A}/H^2 \).

Therefore, by Theorem 3.2, \( T_\varphi \) is Fredholm if and only if \( \varphi \) is invertible in \( C(\mathbb{T}) \). \( \square \)

**Corollary 3.6.** If \( \varphi \in C(\mathbb{T}) \), then \( \sigma_e(T_\varphi) = \varphi(\mathbb{T}) \).

**Proof.** By the Atkinson characterization and Theorem 3.2, \( \sigma_e(T_\varphi) = \sigma(\pi(T_\varphi)) = \sigma(\varphi) = \varphi(\mathbb{T}) \). \( \square \)

Hence, a Toeplitz operator with continuous symbol has connected essential spectrum. In particular, \( \sigma_e(u) = \sigma(T_u) = \mathbb{T} \).

3.2. The Baby Index Theorem. Now, we calculate the Fredholm index of Toeplitz operators \( T_{\varepsilon_n} \).
Since \( \ker u = 0 \) and \( u(H^2) \) is the closed linear span of the set \( \{ \varepsilon_n \mid n \geq 1 \} \), \( \text{ind}(T_{\varepsilon_n}) = \text{ind}(u) = 0 - 1 = -1 \). So, the unilateral shift \( u \) has index \(-1\). If \( n \in \mathbb{N} \), then \( \text{ind}(T_{\varepsilon_n}) = \text{ind}(u^n) = n \text{ind}(u) = -n \), where Fact 3.3 has been used in the second equality. By, the equality \( T_{\varepsilon_n} T_{\varepsilon_n} = I \)
one also infers that \( \text{ind}(T_{\varepsilon_n}) = n = -(-n) \).

We will generalize this result, which is the simplest case of the Atiyah-Singer index theorem on odd-dimensional manifolds.

**Theorem 3.7** (The baby index theorem). If \( \varphi \) is an invertible element in \( C(\mathbb{T}) \), then
\[
\text{ind}(T_\varphi) = -w_n(\varphi),
\]
where \( w_n(\varphi) \) is the winding number of \( \varphi \) with respect to the origin.
The winding number of the continuous function \( \varphi : \mathbb{T} \to \mathbb{C} \setminus \{0\} \) is the unique integer \( n \) given by next lemma.

**Lemma 3.8.** If \( \varphi \) is an invertible function in \( C(\mathbb{T}) \), then there exists a unique integer \( n \in \mathbb{Z} \) such that \( \varphi = \varepsilon_n e^{\psi} \) for some \( \psi \in C(\mathbb{T}) \).

**Proof.** We only show the uniqueness of the integer \( n \) (for the existence of \( n \) and \( \psi \); see the proof of Lemma 3.5.14 of [1]). We need only show that if \( \varepsilon_n = e^{\psi} \) for some \( \psi \in C(\mathbb{T}) \), then \( n = 0 \). The map

\[
\alpha : [0,1] \to \mathbb{Z}, \ t \mapsto \text{ind}(T_{e^{\psi}})
\]

is continuous and has discrete range and connected domain, so it is necessarily constant. Hence, 
\[
-n = \text{ind}(T_{e^{\psi}}) = \alpha(1) = \alpha(0) = \text{ind}(T_1) = \text{ind}(1) = 0.
\]

**Proof of Theorem 3.6.** Suppose that \( \varphi \in C(\mathbb{T}) \) is an invertible element with \( \text{wn}(\varphi) = n \). Now, \( \varphi = \varepsilon_n e^{\psi} \) for some \( \psi \in C(\mathbb{T}) \), and \( T_\varphi = T_{\varepsilon_n} T_{e^{\psi}} \) if \( n \geq 0 \) and \( T_\varphi = T_{\varepsilon_n} T_{e^{\psi}} \) if \( n < 0 \), by Theorem 2.3. Hence,
\[
\text{ind}(T_\varphi) = \text{ind}(\varepsilon_n) + \text{ind}(e^{\psi}) = -n
\]
since \( T_{e^{\psi}} \) is of index zero by the proof of Lemma 3.8.

Since the winding number of \( \varphi \) is unchanged as long as the homotopy class is fixed, so is the "analytic invariant", the Fredholm index of a Toeplitz operator \( T_\varphi \).

**Corollary 3.9.** Let \( \varphi \in C(\mathbb{T}) \) is an invertible element. Then the following conditions are equivalent:

1. \( T_\varphi \) is invertible,
2. \( \text{ind}(T_\varphi) = 0 \),
3. \( \varphi = e^{\psi} \) for some \( \psi \in C(\mathbb{T}) \).

**Proof.** We need only to show the implication (3)\(\Rightarrow\)(1) since the others are obvious from the preceding result.

First, if \( \rho = \sum_{|n| \leq N} \lambda_n \varepsilon_n \in \Gamma \), write \( \rho' = \sum_{n=0}^N \lambda_n \varepsilon_n \) and \( \rho'' = \sum_{n=1}^N \lambda_n \varepsilon_{-n} \). Then, \( \rho = \rho' + \rho'' \) and \( \rho', \rho'' \in H^\infty \). By (1) of Lemma 1.6., it follows that \( e^{\rho'}, e^{\rho''} \in H^\infty \). By using Theorem 2.3, one gets \( T_{e^{\rho'}} \) and \( T_{e^{\rho''}} \) are invertible with inverses \( T_{e^{-\rho'}} \) and \( T_{e^{-\rho''}} \) respectively.

Now, if \( \psi \in C(\mathbb{T}) \) is an arbitrary element, then by the density of \( \Gamma \in C(\mathbb{T}) \), we may choose a trigonometric polynomial \( \rho \) as above such that \( \|1 - e^{\rho'-\rho}\| < 1 \). Then \( T_{e^{\rho}} = T_{e^{\rho'}} e^{-\varepsilon_n \rho''} = T_{e^{\rho'}} T_{e^{-\varepsilon_n \rho''}} T_{e^{\rho'}} \). Since \( \|1 - T_{e^{\rho'-\rho}}\| = \|T_{e^{-\varepsilon_n \rho''}}\| = \|1 - e^{\rho'-\rho}\| < 1 \), the operator \( T_{e^{\rho'-\rho}} \) is invertible. Hence, \( T_{e^{\rho}} \) is a product of invertible operators and is therefore invertible.

We have an application of the baby index theorem to the spectral theory of Toeplitz operator.

**Theorem 3.10.** The spectrum of a Toeplitz operator with continuous symbol is connected.

**Proof.** If \( \varphi \in C(\mathbb{T}) \), then by the baby index theorem and Corollary 3.8,
\[
\sigma(T_\varphi) = \varphi(\mathbb{T}) \cup \{ \lambda \in \mathbb{C} \mid \text{wn}(\varphi - \lambda) \neq 0 \}
\]
Therefore, \( \sigma(T_\varphi) \) is a compact set consisting of the connected compact curve \( \varphi(\mathbb{T}) \) and some of its holes, and therefore \( \sigma(T_\varphi) \) is connected.

As Question 2.5, a natural question arises from this theorem.

**Question 3.11.** Is the spectrum of a Toeplitz operator necessarily connected?

This question was posed by Halmos, and then Harold Widom [6] answered by proving the following theorem.

**Theorem 3.12** (H. Widom). Every Toeplitz operator has a connected spectrum.
3.3. **Coburn’s Theorem.** In this last section, we first prove an important structure theorem for isometries in a $C^*$-algebra, which is an application of the Toeplitz algebra. Then, we consider the universal property of the Toeplitz algebra $\mathcal{A}$ (Coburn’s theorem).

**Remark 3.13.** Since $C(T)$ is a $C^*$-algebra generated by $e_1$, $\mathcal{A}$ is generated by the unilateral shift $u = T_{e_1}$.

**Theorem 3.14** (Wold-von Neumann). If $W$ is an isometry on a Hilbert space $H$, then $W$ is a unitary, or a direct sum of copies of the unilateral shift, or a direct sum of a unitary and copies of the unilateral shift.

**Remark 3.15.** Let $\alpha$ be a cardinal number. We denote by $w^{(\alpha)}$ the direct sum of $\alpha$ copies of the unilateral shift $u$. Theorem 3.13 says that an isometry $W$ is of the form

$$W = u^{(\alpha)} \oplus U,$$

where $U$ is a unitary (possibly vacuous).

**Proof.** We may suppose that $W$ is neither a unitary nor a sum of copies of the unilateral shift. Set

$$K = \bigcap_{n=0}^{\infty} W^n(H).$$

Then, $W(K) = K$, and therefore $K$ reduces $W$. We set $W = w \oplus w'$ where $w$ and $w'$ are compressions of $W$ to $K$ and $K^\perp$ respectively. Since $w$ is an isometry, the equation $W(K) = K$ implies $w$ is surjective, therefore $w$ is a unitary. So, we need only to prove that $w'$ is a direct sum of copies of the unilateral shift $u$.

Now, set

$$L = (W(H))^\perp.$$

Then, $W^n(L) \subseteq W(H) = L^\perp$ for all $n > 0$, so if $m, n \in \mathbb{N}$ and $m \neq n$, then $W^m(L) \cap W^n(L) = \emptyset$. We claim that

$$\bigoplus_{n=0}^{\infty} W^n(L) = K^\perp.$$

Clearly, $W^n(L) \cap W^{n+1}(L) = W^{n+1}(H)$. Since $K \subseteq W^{n+1}(H)$, one infers that $W^n(L) \subseteq K^\perp$, and therefore $\bigoplus_{n=0}^{\infty} W^n(L) \subseteq K^\perp$. To show $\bigoplus_{n=0}^{\infty} W^n(L) \supseteq K^\perp$, it suffices to show that if $x \in H$ such that $x \perp W^n(L)$ for all $n \in \mathbb{N}$, then $x \in K$. Obviously, this is true for $n = 0$. If $x \in W^n(H)$, then there exists $y \in H$ such that $x = W^n(y)$. Since $W^n(y) \perp W^n(L)$, then $y \in L^\perp = W(H)$, and therefore $x \in W^{n+1}(H)$.

Let $E$ be an orthonormal basis for $L$. Since $W$ is an isometry, $\bigcup_{n=0}^{\infty} W^n(E)$ is an orthonormal basis for $K^\perp$. For each $e \in E$, let $L_e$ be the Hilbert subspace of $K^\perp$ having $(W^n(e))_{n=0}^{\infty}$ as orthonormal basis. Then,

$$K^\perp = \bigoplus_{e \in E} L_e,$$

each $L_e$ is invariant for $W$, the compression $W_e$ of $W$ to $L_e$ is the unilateral shift, and

$$w' = \bigoplus_{e \in E} W_e. \quad \square$$

Before going to Coburn’s theorem, we recall a basic tool in the operator theory ([1] section 2.1).

**Fact 3.16** (the functional calculus). Let $B$ be a unital $C^*$-algebra, $a$ be a normal element in $B$ and $z : \sigma(a) \to \mathbb{C}$ be the inclusion map. Then there exists a unique unital isometric $*$-homomorphism

$$\Theta_a : C(\sigma(a)) \to B$$

such that $\Theta_a(z) = a$ and the image $\Theta_a(C(\sigma(a)))$ is the $C^*$-algebra generated by the unit of $B$ and $a$. 
The map $\Theta_u$ is called the functional calculus at $a$. Note that if $f \in C(\sigma(a))$, then we write $\Theta_u(f) = f(a)$.

**Lemma 3.17.** If $w$ is a unitary in a unital $C^*$-algebra $B$. Then there is a unique unital $*$-homomorphism $\varphi : C(\mathbb{T}) \to B$ such that $\varphi(z) = w$.

**Proof.** Observe that $\sigma(w) \subseteq \mathbb{T}$ and that the map

$$i : C(\mathbb{T}) \to C(\sigma(w)), \quad f \mapsto f|_{\sigma(w)}$$

is a unital $*$-homomorphism. One get $\varphi$ by setting $\varphi = \Theta_w \circ i$. Since $z$ generates $C(\mathbb{T})$, $\varphi$ is unique. \hfill $\square$

**Theorem 3.18** (Coburn). Suppose that $w$ is an isometry in a unital $C^*$-algebra $B$. Then there exists a unique unital $*$-homomorphism

$$\Phi : \mathbb{A} \to B$$

such that $\Phi(u) = w$. Moreover, if $ww^* \neq 1$ (that is $w$ is not unitary), then $\varphi$ is isometric.

**Proof.** The uniqueness of $\Phi$ is clear by Remark 3.13. By the GNS-construction, we reduce to the case where $B$ is a unital $C^*$-subalgebra of $B(H)$ containing $1_H$ for some Hilbert space $H$. By the Wold-von Neumann decomposition, we can write

$$H = \bigoplus_{\lambda \in \Lambda} H_\lambda, \quad w = \bigoplus_{\lambda \in \Lambda} w_\lambda,$$

where $H_\lambda$ are Hilbert spaces and each $w_\lambda \in B(H_\lambda)$ is a unitary or a unilateral shift.

If $w_\lambda$ is a unitary, then by Theorem 3.2 and Lemma 3.17 and by composing all the following maps

$$\mathbb{A} \xrightarrow{\pi} \mathbb{A}/K(H^2) \xrightarrow{\Psi^{-1}} C(\mathbb{T}) \xrightarrow{\varphi} B(H_\lambda),$$

where $\pi$ is the quotient map, one gets a unital $*$-homomorphism $\Phi_\lambda : \mathbb{A} \to B(H_\lambda)$ such that $\Phi_\lambda(u) = w_\lambda$.

If $w_\lambda$ is a unilateral shift, then there exists a unitary $U_\lambda : H^2 \to H_\lambda$ such that $w_\lambda = U_\lambda u U_\lambda^*$. Hence, the map

$$\Phi_\lambda : \mathbb{A} \to B(H_\lambda), \quad a \mapsto U_\lambda a U_\lambda^*,$$

is an isometric unital $*$-homomorphism such that $\Phi_\lambda(u) = w_\lambda$.

Now, by preceding paragraphs, we get the family of representations $(H_\lambda, \Phi_\lambda)_{\lambda \in \Lambda}$ of the $C^*$-algebra $\mathbb{A}$. Let $(H, \Phi)$ be the direct sum of $(H_\lambda, \Phi_\lambda)_{\lambda \in \Lambda}$. Then $\Phi : \mathbb{A} \to B(H)$ is a unital $*$-homomorphism such that $\Phi(u) = \bigoplus_{\lambda \in \Lambda} w_\lambda = w$. Moreover, since $\Phi(u) \in B$ and $u$ generates $\mathbb{A}$, one deduces that $\Phi(\mathbb{A}) \subseteq B$.

Finally, we suppose that $ww^* \neq 1$. Then some $w_{\lambda_0}$ is a unilateral shift. Hence, the representation $(H_{\lambda_0}, \Phi_{\lambda_0})$ is faithful, so is $(H, \Phi)$. Therefore, using the fact that an injective $*$-homomorphism between $C^*$-algebras is necessarily isometric, $\Phi$ is isometric.

\hfill $\square$

**References**


