Nagoya University, G30 program Fall 2015

Mathematics Tutorial Ia (calculus) Instructor : Serge Richard

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Homework 7

Recall that for any $a \in \mathbb{R}_+^*$ and for any $x \in \mathbb{R}$ the following equality holds:

$$\ln(a^x) = x \ln(a).$$

Based on this equality, one sets

$$a^x \equiv \mathrm{e}^{\ln(a^x)} := \mathrm{e}^{x \ln(a)},$$

with $y \mapsto \mathrm{e}^y$ the unique function satisfying $\big(\mathrm{e}^y\big)' = \mathrm{e}^y$ and $\mathrm{e}^0 = 1$, *cf.* Exercise 5 in Homework 2. It follows from this definition that for any $x \in \mathbb{R}_+^*$ and $r \in \mathbb{R}$ one has

$$x^r = \mathrm{e}^{\ln(x^r)} = \mathrm{e}^{r \ln(x)}.$$

Exercise 1 *If we set* $\mathrm{e}^1 := e = 2.718...$, *check that* $\ln(e) = 1$ *and* $e^x = \mathrm{e}^x$.

Exercise 2 *For any* $a \in \mathbb{R}_+^*$, *show that* $\lim_{x\to 0} \frac{a^x - 1}{x} = \ln(a)$. *By taking Exercise 1 into account, show that*

$$\lim_{x\to 0} \frac{\mathrm{e}^x - 1}{x} = 1.$$

Exercise 3 *Compute the following limits:*

1. $\lim_{x\to 0} \frac{\ln(1+x)}{x}$,
2. $\lim_{x\to 0} (1+x)^{1/x}$,
3. $\lim_{x\to \infty} \big(1 + \frac{1}{x}\big)^x$,
4. $\lim_{x\to \infty} \big(1 + \frac{r}{x}\big)^x$, *for any* $r > 0$.

Exercise 4 *Compute the derivative of the following functions:*

$$f : \mathbb{R} \ni x \mapsto a^x \in \mathbb{R} \ \text{ for any } a > 0, \qquad g : \mathbb{R}_+^* \ni x \mapsto x^x \in \mathbb{R}.$$

Exercise 5 *Sketch the graph of the function* $f : \mathbb{R} \ni x \mapsto x\mathrm{e}^x \in \mathbb{R}$.